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THE EXACT DESIGN OF STATICALLY INDETERMINATE FRAMEWORKS. AN EXPOSITION OF ITS POSSIBILITY, BUT FUTILITY.

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WITH DISCUSSION.

INTRODUCTION.

In the theoretic consideration of all material systems it is customary, in fact necessary, to replace them by more or less equivalent but simpler systems. This alone makes their analytic examination possible, not to say practicable; and the results thus obtained, although applying strictly only to the ideal constructions of theory, throw much valuable light on the workings of the actual material systems.

But so accustomed have we become to this procedure that we often forget that between the theoretic results and the actual conditions there must always exist a divergence, greater in proportion as the difference between the imaginary system and the real system is the greater. This neglect leads to two serious faults: *First*, in the consideration of existing systems, replacing them by theoretic systems, which insufficiently represent them, and applying the conclusions thus derived as though they were practically exact: *Second*, in the

creation of new systems, constructing them so differently from the ideal systems on which they are based that the actual conditions are quite unlike those supposed. This has given rise to many strange fallacies, has held back practice and has discredited theory.

Frameworks furnish excellent illustrations of this, for the ideal framework has always been a far simpler structure than the real framework, and its theory correspondingly different from actual conditions. Whereas the ideal framework consists of a group of straight members so joined at their ends as to move freely about the points of connection, loaded only at these points, and—neglecting the effects of dead weight and wind pressure—subject, therefore, only to direct stress of tension or compression in the members, actual frameworks always have joints more or less stiff, their members are frequently joined at other points than their ends, and they are loaded at other points than their joints, thus introducing flexure of the members, as well as direct stress, as a result of the application of loads.

But it has so long been customary to overlook these divergencies between theory and practice, to gratuitously assume them of no consequence, that we now, as a matter of course, figure our simple theoretic system and then apply the results to our complex actual system, without a doubt, apparently, that this application is not only justifiable, but practically exact. This has resulted in the perpetuation of defective construction and the fallacy that increased strength and stiffness, with the same amount of material, is to be obtained by stiff joints, broad members rigidly attached where they pass one another, continuous members, etc. To this day, some eminent engineers uphold these as virtues. And, on the other hand, this has resulted in indifference as to the variations between theory and practice which actually exist, and a neglect to develop, and still more to apply, more exact parallelism, both in theory and practice. In America a partial attempt to fulfill these suppositions of the ideal framework by the use of pin connections was made long ago, yet we continue to use continuous chords, and floor systems, and many riveted connections of which the theory takes no account. In Europe a few attempts have been made to calculate closer theoretic parallels to actual structures, and these more exact calculations have shown how very considerable were the neglected elements of stress and the importance of eliminating these "secondary" stresses as they are called; but these calcula-

tions are very recent, and are extremely few in number, for they are very laborious.

The net result is, that not only do theory and practice diverge to an important extent, but that this divergence is given practically no attention. This is not as it should be, and we may look forward to considerable improvement in this respect in the near future. The writer believes that a much closer approach of the actual to the ideal framework is not only practicable, but desirable; and, on the other hand, he considers more exact calculation, that is, calculation of the structures more nearly as they are, a requisite to progress. Expensive mechanical devices should not be used in the effort to approach more nearly ideal construction where no corresponding practical advantage can be shown, and laborious calculations are out of place where the limits of error of simple approximate calculations are known and can more cheaply be taken into account through liberality in construction. But what may be insisted on is that either the actual conditions shall be taken into account through more exact calculation, or that no pretence of accuracy shall be made where there has been none, and that the divergences between the theory used and the construction actually carried out shall be allowed for liberally.

The object of this paper is not to consider the more exact calculation of actual structures, but rather to examine certain questions in connection with ideal frameworks, of the greatest practical importance because at the foundation of actual designs. It deals with statically indeterminate ideal frameworks, that is, ideal frameworks as previously defined, in which not all the members are essential, or, in other words, in which some members are superfluous. What are the conditions involved and how may such frameworks be exactly designed; what are their qualities, and what their merits and demerits? Such is the matter which this paper seeks to consider. Its purpose is not to give instruction in methods, but simply to indicate the principles involved, to point out some fundamental but not generally known facts, and to discuss broadly, with the aid of a few simple illustrations, the bearing of these facts on practice. And let not such discussion of ideal construction be underrated and regarded as of no value in connection with actual structures, which all, more or less, diverge from the ideal, for it must be remembered that the ideal here considered is the basis on which all actual practice is founded.

The use of statically indeterminate frameworks, in the sense here mentioned, has been exceedingly common in Europe, where it has been, and is even now, advocated extensively by the most eminent engineers. Continuous bridges, two-hinged and hingeless arches, arches supporting continuous trusses, and their inversions of suspension type, not only are met frequently in existing structures, but are used frequently in new projects. And there are indications that here in the United States these constructions are gaining a certain hold, as shown by the erection of certain recent arches and suspension bridges, and above all by the proposed huge suspension bridge at New York. Moreover, since most frameworks are actually space-frameworks, formed by uniting several plane frameworks, and this connection is made without consideration of statical determination, most frameworks—as bridge trusses with their connecting wind bracing—are actually indeterminate, whether or not their component plane frameworks are statically determined when alone. This gives great prominence to the subject of the construction of frameworks with superfluous bars, and makes a clear understanding of them exceedingly important. Is it advisable to use such structures rather than the much simpler "just determined" construction generally advocated in America? Which is the better, and why?

In the course of the more mathematical examination which will follow, the answer to this question will present itself, but it is advisable here to review briefly certain facts.

European engineers have always been the principal advocates of indeterminate construction. They have insistently claimed for it the virtues of greater economy, strength, safety and stiffness. Practically, no demonstration of these claims has been forthcoming, except, perhaps, proof that the continuous girder possessed these advantages as compared with equal free spans; but this comparison was unfair, and the proper rejoinder to it was eventually given by the cantilever. Grudgingly conceding its equivalence, European engineers continue to claim, without demonstration, the superiority of the remaining types, particularly of the two-hinged and hingeless arches over the three-hinged arch.

These claims have not been wholly uncontested. As early as 1876, Maurice Levy, in Paris, insisted strongly on the universal economic superiority of frameworks with no superfluous bars. He supported his contention by general reasoning, and demonstrated

mathematically, that in a certain exceptional case with superfluous bars the economy was no greater than without such bars. But little attention seems to have been paid to Levy's statements, possibly because they were not accompanied by any general mathematical proof. The subject, from this point of view, seems to have been dropped entirely from consideration, and the writer, in his studies in Paris, Zurich and Berlin, perhaps the most notable of the European engineering schools, has found no further mention of it. In these schools statically indeterminate structures are universally taught and advocated, being regarded, apparently, as more difficult, but better, than the simpler, determined structures. Beautiful methods of calculation have been developed for the more complex structures, and the professors, possibly fascinated by the charms of these computations, gave them great prominence, and fostered among the students a predilection for difficult types, without making any study of the question, whether or not there were real advantages in their use to offset their difficulties.

Feeling convinced that, while the knowledge of methods of calculation of difficult structures was exceedingly valuable because of their frequent occurrence, and because of unavoidable occasions for their use, the encouragement, even preference, given to the use of such construction was mistaken, the writer, fully informed as to European ideas and methods, undertook a general investigation of the comparative merits of frameworks with and without superfluous bars. The result was the discovery of some curiously unknown, although most fundamental, facts concerning the design of frameworks with superfluous bars and the distribution of stresses therein, together with a somewhat general demonstration of the economic superiority of statically determined over indeterminate construction. In a simple and popular article published in the *Technology Quarterly*,* for June, 1897, these results were set forth, but it was desired to present the matter, in a somewhat more mathematical and complete form, directly to the engineering public, for whom it would have a particular interest.

Consequently, the present article has been written. In it the simple illustrations and much of the matter of the preceding article have been passed over with a mere reference, the subject being treated somewhat more generally, and much space has been devoted to some illustrations of selected, extremely simple cases of exact design, with

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the purpose of demonstrating the impracticability, the futility, even the folly of attempting such designing in practice, rather than furnishing instruction in its use. It is desired simply that those who read this, by the inspection of the amount and difficulties of the work as exposed in the illustrations, may form some adequate notion of why the exact design of frameworks with superfluous bars is not to be thought of, and, therefore, why such design in practice, in the future as in the past, will be made by methods of trial, which can only give at best, after infinite labor, most imperfect and uneconomic results. But it is desired that they may examine with care the results obtained in the illustrations, to convince themselves that even with exact methods, giving the most perfect possible designs, and when neglecting temperature effects and many other serious objections, the use of superfluous bars furnishes no sensible advantage, while, including these inevitable accompaniments of indeterminate construction, it is in all respects inferior to the simple determinate construction, taught and advocated in our schools, and with which American bridge engineers are perfectly familiar.

To those whose interest and leisure permit, more careful perusal of the illustrations is recommended, and in particular the working out of similar examples. Such work would form a valuable supplement to the little that the writer is here able to offer. And further, he would suggest the careful analysis of some notable existing indeterminate structures, and their comparison with designs of parallel statically determined structures. The results will inevitably be surprising in their demonstration of the sad deficiency of such indeterminate structures, judged by the standards of good proportioning insisted on in all statically determined designs. American engineers may in the future devote their attention to the determination of the best statically determined forms, to the development of statically determined space-frameworks and to the perfection of details intended to ensure the usually assumed ideal simplicity of stress, with the assurance that they are on the sole road to the best designs.

The writer further ventures to hope that structures now in contemplation but not commenced, projected in statically indeterminate form, will be reconsidered, and that more economic statically determined forms, which certainly exist, will be determined and used in their stead.

THE EXACT DESIGN OF STATICALLY INDETERMINATE FRAMEWORKS.

In an ideal framework, since it is composed only of members joined at their ends, freely movable at these joints and loaded only at the joints, we are concerned only with the equilibrium of forces acting at points—the joint centers. Therefore, the conditions at each joint are three in number, that the resultant forces along three rectangular axes shall be zero. If a framework have n joints this furnishes $3n$ equations of condition which must be satisfied. Of these $3n$ equations but $3n - 6$ are independent equations involving the inner forces alone, since they must together satisfy six conditions of equilibrium among the outer forces acting on the framework, including in these outer forces the elements of the reactions as well as the loads. In general, for the stability of the structure, these elements of the reactions must be at least six in number. Let their number be r, and the number of members of the framework or bars be m. Then, for stability, $m + r$ must at least equal $3n$. But $m + r$ may perfectly well exceed $3n$, in which case the framework becomes statically indeterminate, because the $3n$ conditions furnished by statics are insufficient to determine the $m + r$ unknown bar stresses and reaction elements.

If the framework were statically determined we should have $m + r = 3n$, the form of the framework would be just determined through the lines of its figure, it would be "just stiff"; remove any bar and it would become a mechanism; alter slightly the length of any bar or the position of any support and the figure of the framework would be altered slightly without further consequences; no stress would exist in the members, except through loads; temperature changes, whether uniform or varying, would change slightly the form of the framework without affecting it otherwise; the stress in each member due to any load would be calculable easily and without regard to its own material and its elastic qualities or those of its neighbors; each member could be designed readily and exactly under any given series of loadings to satisfy almost any given conditions, for the dimensioning of each member would be practically independent of that of all others; and, finally, all displacements of the joints of the framework due to stresses or temperature changes would be just defined through the corresponding changes of the lengths of the members, and would be easily calculable.

If to such a framework we were to add further bars, the joints

remaining as before, so that $m + r > 3n$, the framework would become overdetermined; some of the bars could be removed, and the framework would still be stiff; alter slightly the length of any of these bars or the position of a support, and the framework would be put under stress; stress without loads might, therefore, exist, and temperature changes might cause very considerable stresses; the stresses under given loads could only be calculated by long and laborious application of the equations of elasticity, in which not only the dimensions of each member but its material and elastic qualities would have to be taken into consideration; the stress in a single member could not be determined independently of other members; the design of a given member might be affected greatly by the design of many other members, and it would be very difficult, if not impossible, to satisfy accurately given conditions under a series of loadings; finally, minute changes in the lengths of some of the bars, or slight yielding of the supports, might change materially the distribution of stress, so that it would be exceedingly difficult in practice to obtain even approximately any stress distribution which in the design was determined upon, and to maintain such stress distribution in the face of any slight yielding of joints and supports due to changing loads, climate and other external, or, perhaps, also internal, causes.

Such are the structures whose exact design we are about to consider. In the foregoing we have regarded them as formed by the addition of further members to "just determined" frameworks, although they have rarely been thus conceived. This manner of regarding them is most serviceable in connection with their calculation, but the course of evolution has rather been the derivation of "just determined" forms from pre-existing indeterminate forms through the dropping of superabundant (or superfluous) members.

The recital of the more obvious characteristics of the indeterminate frameworks, just made, would seem alone a sufficient condemnation of their use in practice, and further facts, developed in the later analysis, will serve to confirm definitely this view. The statical analysis of frameworks with superfluous bars shows them to be inferior in economy and strength while not superior in stiffness to the best parallel designs without superfluous bars.

The objectionable characteristics of indeterminate structures have long been known, but hitherto have been regarded as offset by certain

supposed virtues of economy and stiffness which have been ascribed to them. Yet the correction of these mistaken beliefs through analysis, which is the object of this paper, will destroy the prestige of indeterminate structures only in part. They have, and will long continue to have, the support of many able engineers, whose experience, having been largely with such structures, prejudices them in their favor and disables them from appreciating and weighing their defects clearly. Following the common human device of seeking in reason justification for beliefs rather than through reason establishing belief, some of these engineers will make much of what analysis has not yet defined and will strive to make of the unknown, and, for the present, perhaps, unknowable, not simply a defence, but a justification of the continuance of their beliefs. We shall find them taking to ground so difficult that the theorist, at least as yet, is unable to follow and settle the matter decisively, and maintaining that, in general, the problems connected with frameworks are not static, but dynamic; that inferiority, demonstrated for static loads, no longer is true of moving loads; and that the introduction of vibration in addition to static deflections wholly alters the case and leaves indeterminate structures, after all, masters of the field.

But it must be remembered that these will only be claims, not proofs. They can only be supported by general reasoning, and can be effectively answered in the same fashion. Experience has not demonstrated that dynamic defects are necessary accompaniments of static determination, and all not proven by analysis remains disputable. Let us only not be outfaced by displays of erudition or assertions of authority, and we will speedily perceive that in appeals to unconsidered and complex phenomena (the writer has heard brought up the speed at which stress travels and the consequent effect of the time of loading on stress distribution), we have only to do with appeals to ignorance, which can hardly be regarded as adding strength to the arguments of those who make them.

The writer does not wish by the preceding to belittle the importance of dynamic phenomena. In due time and in their order they should be and doubtless will be studied and given proper consideration. But first we must study and understand the static phenomena which are at their foundation, the truths of which they will not change but only add to.

In the following paper no attempt is made to treat mathematically

other than the static side of the question. And to begin with we may point out a fact, hitherto almost wholly disregarded, and which makes indeterminate frameworks worthy of the name far more than does the fact that they are statically indeterminate.

We have already noted that in indeterminate frameworks the number of unknown quantities is greater than the number of equations of condition furnished by statics, that is $m + r > 3n$. We may conveniently regard the framework as possessing as many superfluous bars as $m + r$ exceeds $3n$. Let the bars i ($i = h$ to m inclusive) be selected as the superfluous bars; then the remaining bars e ($e = 1$ to g inclusive) form with the supports a statically determinate framework. Let S_{e_0} be the stress in any bar e of this statically determined framework if this framework alone carried the loading, let $S'_e{}^{(i)}$ be the stress in this bar e which a tension of unity in one of the superfluous bars i would cause in e if there were no external load and if of the superfluous bars the bar i alone were in action, and let S_i be the actual stress in the superfluous bar i when the given load is applied to the actual framework with all its superfluous bars acting. Then the actual stress S_e in a bar e is evidently given by the formula:

$$S_e = S_{e_0} + \sum_{i=h}^{i=m} S_i S_e'^{(i)}$$

Now—and here is the commonly neglected consideration—as far as the laws of statics are concerned, only the S_{e_0} and the $S_e'^{(i)}$ are fixed and determinate, while the S_i may be anything; they are truly indeterminate and not, as commonly stated, fixed by certain auxiliary equations of elasticity. What does fix them in a structure with given elastic constants and given bar sections is, under given conditions of temperature, the precise lengths of the bars. But, as a matter of fact, in any existing structure these lengths of bars are never known with anything like exactitude, as will be evident when one considers that, with a modulus of elasticity of 30 000 000 lbs. per square inch, a bar of steel 10 ft. long, whose length is not known within a thousandth of a foot, would not have its intensity of stress known within $\frac{30\,000\,000}{10 \times 1\,000} = 3\,000$ lbs. per square inch, or within about a third of its probable working stress. Designers and computers have passed over this difficulty, often, if not generally, unconsciously, by the simple assumption, unwarranted by any corresponding precautions in construction or knowledge of existing structures, that, at a certain

temperature and with a certain loading, a certain state of stress exists throughout the structure. Usually, this takes the simple form of the assumption that the structure, when free from all loading and uniformly at the mean temperature, is free from all stress. Then truly the equations of elasticity define the stresses under all other conditions of loading and temperature, but it is this far from warranted assumption and not the equations of elasticity which removes the indetermination.

Since the stresses in the superfluous bars are not fixed by equations of elasticity, but by their lengths, we evidently have it in our power—in designing at least—to give these bars any stresses we please under a given loading by giving them suitable lengths, and this without in any way altering the figure of the framework (because the changes of length of the superfluous bars required in all practical cases would be very small), and without altering the sections of the bars, that is, the design as it is ordinarily defined. If these superfluous bars be k in number this gives us a k^{tuple} degree of freedom in the design of such a framework, and the same degree of indetermination in the computation of such an existing framework. Unlike the case of a statically determined framework where the distribution of stresses due to a given loading is invariable, that distribution may here be varied in an infinity, a k^{tuple} infinity, of ways. In designing, we may select any one of these at will, then design the cross-sections to suit, and finally compute very exactly the corresponding lengths of the bars. It will remain to construct them exactly as computed to realize the designed stress distribution. But if the structure be an existing one given us to compute, who can say which one of this infinity of possible stress distributions is that actually existing under given conditions? The problem is actually indeterminate and in increasing degree as the number k of superfluous bars is greater.

We are now prepared to consider the subject of the title, "The Exact Design of Indeterminate Frameworks." But first, what is meant by "exact" design? Any one familiar with the design of indeterminate frameworks is aware that, unlike the design of determinate frameworks (if we neglect the estimation of dead weight), the process is essentially tentative instead of direct. That is, given the figure of the framework, certain bar sections are assumed, calculations of the stresses corresponding made (on the assumption of no stress under no load at mean temperature), the divergences from the desired con-

ditions as to intensity of stress noted, proper corrections of the sections guessed at, the computations all repeated, new corrections guessed at, then new computations made, and so on, repeating again and again all the work until the result approaches that desired sufficiently to be acceptable. But as an alteration in the section of any one bar affects the stress not in itself alone, but in many other bars also, this process of correction is not simply exceedingly laborious, but, as every designer of such structures knows, very uncertain; and after infinite labor the result obtained is usually one, the deviations of which from the set conditions of maximum allowable intensity of stress are such as would not be tolerated by any good engineer in a statically determined framework.

By "exact" design is meant: Expressing in equations the conditions as to intensity of stress to be satisfied by the solution under the given loadings, and then by direct solution of these equations, without any tentative process (other than that mathematically necessary to solve the equations), obtaining at once and exactly the desired design, *e. g.*, the cross-sections and the bar lengths. It will be appreciated that such an exact method, apart from any question of practical advantage over the old tentative method, would be likely to throw most instructive light on the properties of indeterminate designs, which the old method could only feebly give through statistics, or practically not at all; and the object in bringing forward this "exact" method is simply to throw such light; for, apart from the fact that the writer reaches through this method the conclusion that the best designs of indeterminate frameworks are necessarily inferior to those of determinate frameworks which necessitate no such methods, the illustrations to follow will, among other things, show conclusively that the mathematical difficulties of the exact method render its application impracticable in general.

To begin with, we will consider the simple but very instructive and important case of design for a single given loading, subject to the condition of a given intensity of stress in each bar under that loading. This case will be somewhat briefly treated, since, in an earlier article,* it has been fully considered and illustrated.

As pointed out previously, an infinite variety of stress distributions under a given loading are all equally possible, regardless of the cross-

* *Technology Quarterly*, June, 1897. "Some Fundamental Propositions Relating to the Design of Frameworks."

sections of the members. Having selected at will any one of these stress distributions, we next design the cross-sections of the various members to carry the stresses thus assigned them with the assigned intensities of stress, using column formulas or not as we choose. It only remains to determine the lengths of the bars necessary to ensure the supposed stress distribution. This we may do as follows:

In the given figure of the framework the exact length of every line is geometrically determinate if the data for the figure are sufficient, and this we suppose. Thus, to start with, we have what may be called the figure length L_f (where $f = 1$ to m) of every one of the m bars. Now, we know that, supposing the points of support fixed, the exact position of each joint is geometrically determined by the exact lengths from center to center of joints of the series of bars e ($e = 1$ to g), which together form a statically determined framework with the same joints (see page 573). The lengths of the bars i ($i = h$ to m), which we have termed the superfluous bars, are therefore geometrically deducible from the lengths of the bars e. Let us assume the actual unstrained lengths from center to center of joints of the bars e, at standard temperature (lengths which we may in general denote by l), to be the figure lengths, or that $l_e = L_e$. On this basis we now have to determine the actual unstrained lengths l_i from center to center of joints, at the given standard temperature, of the superfluous bars i, so that, under the given loading, each bar shall have the stress we have already determined for it.

These lengths l_i will not in general be the same as the figure lengths L_i, which we may regard as exactly defined through the L_e and therefore known. For, under the given loading the bars e are subject to the stresses S_e and, therefore, no longer have the lengths $l_e = L_e$, but have the strained lengths $L_e + \frac{S_e l_e}{E_e A_e}$, where E_e and A_e are the modulus of elasticity and the section area, respectively, of the bar e. And the distances from center to center of the joints connected by the superfluous bars i, that is the strained lengths of the superfluous bars i, are at the distances defined by and resulting from these strained lengths $L_e + \frac{S_e l_e}{E_e A_e}$. If we determine these strained lengths of the superfluous bars and then design their actual lengths l_i, so that the strained lengths shall correspond to their supposed stresses S_i, that is, so that the strained lengths shall be $l_i + \frac{S_i l_i}{E_i A_i}$ then we shall evi-

dently have so made our design as to secure our supposed distribution of stress. These strained lengths of the bars i might be directly determined by ordinary geometrical principles from the strained lengths $L_e + \frac{S_e l_e}{E_e A_e}$ of the bars e, but, since we have all the figure lengths L, and we may suppose the changes from these to be very small, it is preferable to proceed as follows, making use of the principle of virtual work. Let the bars e suffer any small elongations Δ_e from any causes whatsoever, then the consequent approach of any pair of joints not connected by a bar e will evidently be the same as the work that would be done by two equal and opposite forces of unity, each acting at those joints toward each other, through this approach, and this external work of the pair of forces unity would in turn be equal to the internal work the stresses balancing them in the bars e would do through the elongations Δ_e of the bars e. Now, these stresses in the bars e, balancing such a pair of equal and opposite forces of unity between the joints to be connected by the superfluous bar i, we readily see are the quantities $S'_e{}^{(i)}$, already noted on page 362, so that the approach of the joints to be connected by the bar i, due to the elongations Δ_e of the bars e we at once see would be

$$\delta_i = \sum_{e=1}^{e=g} S'_e{}^{(i)} \Delta_e *$$

In our case, the Δ_e we suppose to be the elongations due to the stresses S_e, that is, we have $\Delta_e = \frac{S_e l_e}{E_e A_e}$, whence follows:

$$\delta_i = \sum_{e=1}^{e=g} \frac{S'_e{}^{(i)} S_e l_e}{E_e A_e}.$$

But $L_i - \delta_i$ is the strained length of the superfluous bar i which is also, as we have noted, $l_i + \frac{S_i l_i}{E_i A_i}$, so that we find for our desired unstrained length of the superfluous bar i the value

$$l_i = L_i - \left(\sum_{e=1}^{e=g} \frac{S'_e{}^{(i)} S_e l_e}{E_e A_e} + \frac{S_i l_i}{E_i A_i} \right)$$

$$= L_i - \sum_{f=1}^{f=m} \frac{S'_f{}^{(i)} S_f l_f}{E_f A_f}$$

(where we must note that the $S'_f{}^{(i)}$ are all zero for all values of f from h to m except i, when $S'_i{}^{(i)} = +1$).

* For a much fuller account of the principle here used and its applications, see the article "On the application of the Principle of Virtual Velocities to the Determination of the Deflections and Stresses of Frames," by Professor George F. Swain, in the *Journal* of the Franklin Institute, April, 1883.

The foregoing neglects any temperature changes, but were there such they would be accounted for by putting $\Delta_f = \frac{S_f l_f}{E_f A_f} + \tau_f t_f l_f$, where τ is the coefficient of expansion and t the increase in temperature above the standard used.

This determination of the exact values of the bar lengths, which ensures the supposed stress distribution under the given loading, completes the design, but there still remains a matter of interest for consideration, the "primary" stresses or stresses remaining in the bars after the loading is removed. That such primary stresses will in general exist is evident from the fact that the superfluous bars have lengths l_i differing more or less from the figure lengths L_i, which would correspond to a condition of no stress at standard temperature with no load. Let us denote the primary stresses by (S); then, reasoning as before, the primary stresses (S_e) in the bars e determine the distance between joint centers of any one of the superfluous bars i to be

$$L_i - \sum_{e=1}^{e=g} \frac{S'_e{}^{(i)} (S_e) l_e}{E_e A_e}.$$

But this is the length of the bar i under its primary stress, or the length $l_i + \frac{(S_i) l_i}{E_i A_i}$; whence we obtain the equation

$$l_i + \frac{(S_i) l_i}{E_i A_i} = L_i - \sum_{e=1}^{e=g} \frac{S'_e{}^{(i)} (S_e) l_e}{E_e A_e}$$

which, adopting the previous notation, we can express in the form

$$\sum_{f=1}^{f=m} \frac{S'_f{}^{(i)} (S_f) l_f}{E_f A_f} = L_i - l_i.$$

Now, for each of the k superfluous bars i we can write out such an equation, each of which contains the primary stresses (S) as the only unknowns, and they appear only in the first degree. But the primary stresses (S_e) in the bars e we know are expressible in terms of the primary stresses (S_i) in the superfluous bars i through the linear equation $(S_e) = \sum_{i=h}^{i=m} (S_i) S_e'^{(i)}$ (see page 362), so that, making these substitutions in the k preceding equations, they become simply k linear equations between the k unknown primary superfluous bar stresses (S_i). These equations solved, from the now known (S_i) we determine the (S_e), and so all the primary stresses. We have only to note that the primary stresses thus found must be less than those occurring under the given loading, for the design to be acceptable.

So much for the design; now consider certain consequences. Suppose the section areas were designed by the simple rule $A = \frac{S}{\sigma}$ where σ was the working stress allowed. Then A being simply proportional to the stress S, each section area, and, therefore, the volume of material of each bar, would be simply proportional to its stress under the given load, and the total volume would be expressed simply by

$$V = \sum_{f=1}^{f=m} \frac{S_f}{\sigma_f} l_f$$

But the stresses S_e in the bars e are expressible in terms of the stresses S_{e_0}, (which would occur in them under the given loading if there were no superfluous bars), and of the arbitrarily variable superfluous bar stresses S_i, through the linear equation $S_e = S_{e_0} + \sum_{i=h}^{i=m} S_i S_e'^{(i)}$ (see page 362), so that ultimately the volume is expressible in the form

$$V = C + \sum_{i=h}^{i=m} c_i S_i$$

where c_i and C are constants. Now, were c_i and C always the same, whatever the S_i might be, V would take all values from $+\infty$ to $-\infty$ by suitably varying the S_i. But, actually, we are limited by the condition that $\frac{S_f}{\sigma_f}$ must always remain positive, so that, if by any variation of the S_i the stress in any bar becomes zero and changes to stress of the opposite kind, its working stress σ at the same instant is changed in kind, which not only involves a change in the sign of σ, but also a change in its value if different working stresses in tension and compression be used. Thus, V is expressed by a discontinuous linear function of the k independent variables S_i, the discontinuity of which occurs only when some bar stress becomes zero. Such a function has a minimum only at a point of complete discontinuity, that is, where the variation of the arbitrary stresses S_i causes k bars to pass simultaneously through zero stress, whence the conclusion:

The least material is required in a framework of given figure to resist a given loading with prescribed intensities of stress in its members, for some distribution of stress which makes zero the stress in as many bars as are superfluous.

That is:

A statically determined framework of included figure is the most economic form of a framework of given indeterminate figure for the support of a given loading.

A simple and much more objective proof of this same fact, on the same basis, due to Professor George F. Swain, will be found in the article in the *Technology Quarterly*, before mentioned. But, it may be objected that both these proofs are based on the simple rule of design $A = \frac{S}{\sigma}$, whereas posts are usually designed by more complicated formulas. Now, it can easily be shown that the change in volume of a given framework due to a slight change $d S_i$ in the stress in one of the superfluous bars i of the framework, independently of all others, is expressed by

$$d V = d S_i \sum_{f=1}^{f=m} \frac{S'_f \phi_f}{\phi'_f (A_f)} l_f$$

where $S_f = \phi_f (A_f)$ expresses the relation between the section area of a bar and its stress and ϕ' is the rate of change of ϕ. Now, providing that $\phi'_f (A_f)$ in no case diminishes numerically as A increases, that is, providing that a post never gains less in strength for a given increase in its area as the area increases, $d V$ will not change its sign from negative to positive during continuous variation, and therefore changing S_i so as to make $d V$ negative, that is, so as to diminish the total volume of material employed, will continue to diminish that volume until a point of discontinuity is attained where some bar's section is reduced to zero. As before, we reach the conclusion that as many bars as are superfluous must have their sections reduced to zero for most economic proportions. But the condition as to post formula on which this conclusion is founded is true of all formulas of the Gordon-Rankine type, those almost universal in practice; therefore, the conclusion as to the superior economy of an included statically determined design holds, under the present standard rules of design.

We may note a further consequence, relating to stiffness. Any change in the positions of the joints, therefore the deflection of any framework, is completely defined by the strains of the bars left after excluding in any way as many bars as are superfluous (see page 365). That is, the deflection of any indeterminate framework from a state of no stress under no load (the common supposition), to its actual state of stress under a given load, is precisely the same as that of any of the determined frameworks of included figure, which we may design to carry the same loading with the same intensities of stress in the corresponding bars. But at least one of these latter frameworks would

carry this load with less material in its bars (see page 368), and therefore be stiffer construction, since, with the same amount of material in its bars, its deflections would be smaller.

Thus we see that a given loading can always be carried, with precisely the same deflections, with the same intensities of stress and with less material in the bars, by a statically determined framework the figure of which is included in that of the given indeterminate framework, than by the given framework, which may be any whatever. These same considerations, by simple modification, yield the same result as applied to the cost of the material where different materials are used, as steel cable for tension members and rolled metal for compression members.

What of the bearing of these facts? While it is true that they hold strictly only for the case of a single loading, their weight for all cases is very considerable.

"Frequently, perhaps usually in practice, some one loading or combination of loadings determines the dimensions of almost all the principal members of the framework, the members not thus determined affecting the total quantity of material employed but slightly. This is particularly the case for large structures in which dead weight plays a leading part, and where the total quantity of material employed is so large that economy in this direction is particularly desirable. In such cases one loading virtually rules the economic design, and our foregoing results hold approximately true. *The consideration of the effects of changes in loading in these cases,* although of great importance as far as the strength of the structure goes, *is of wholly secondary significance from our economic point of view.*"

With this quotation from the article in the *Technology Quarterly*, where all the foregoing will be found more fully explained as well as illustrated, we will turn to the more general and new part of the subject, the exact design of indeterminate frameworks subject to several different loadings.*

Perhaps the most readily comprehensible derivation and most simple form of the equations whose solution furnishes an exact design of any indeterminate framework of given figure subject to any given series of loadings and under any given conditions of design, is the following: As we have already noted on page 366, if, under any loading, 1, the

* The writer here desires to state his indebtedness to Professor George F. Swain for the suggestion of the possibility of such exact design, and, further, to acknowledge many other valuable suggestions, as well as a critical review of the present article, by Professor Swain, under whom the writer first studied structures and learned to appreciate the superiority of statically determined construction.

stresses in the members be denoted by S_{f_1} then the unstrained length of any superfluous bar i is shorter than the length of its line in the given figure of the framework by

$$\sum_{f=1}^{f=m} \frac{S_{f_1} \; S'_f{}^{(i)} \, l_f}{E_f \, A_f}$$

And if under any other loading, 2, the stresses in the same framework are S_{f_2} then the same quantity is given by

$$\sum_{f=1}^{f=m} \frac{S_{f_2} \; S'_f{}^{(i)} \, l_f}{E_f \, A_f}.$$

But since the amount by which the unstrained length of any superfluous bar is shorter than the length of its line in the given figure of the framework is not, by supposition, subject to change, through change of loadings (or rather, of the stresses existing under them), we will have, for each superfluous bar i, an equation of the form

$$\sum_{f=1}^{f=m} \frac{S_{f_1} \; S'_f{}^{(i)} \, l_f}{E_f \, A_f} = \sum_{f=1}^{f=m} \frac{S_{f_2} \; S'_f{}^{(i)} \, l_f}{E_f \, A_f}$$

or, in briefer form,

$$\sum_{f=1}^{f=m} \frac{(S_{f_1} - S_{f_2}) \, S'_f{}^{(i)} \, l_f}{E_f \, A_f} = 0$$

Now, whatever our rules of design may be, the section areas A_f will be definite functions, of some sort, of the stresses S_f; and the stresses S_f are in turn linear functions of the stresses S_i in the superfluous bars through the equations

$$S_{f_1} = S_{f_{o_1}} + \sum_{i=h}^{i=m} S_{i_1} \, S'_f{}^{(i)} \qquad S_{f_2} = S_{f_{o_2}} + \sum_{i=h}^{i=m} S_{i_2} \, S'_f{}^{(i)}, \text{ etc.},$$

where the $S_{f_{o_1}}$ are the stresses that load 1 would cause in the statically determined framework composed of bars 1 to g and the $S_{f_{o_2}}$ are the corresponding stresses which load 2 would cause in that framework. Thus the equations shown are implicit equations of necessary relations between the stresses S_{i_1}, S_{i_2}, etc., in the superfluous bars i under the various loadings 1, 2, etc. For each superfluous bar i and for each pair of loadings there is one such equation, that is, if there be N different loadings, for each bar i there will be $N-1$ such

independent equations,* and if there be k superfluous bars in all, this will give a total of $k\ (N-1)$ such independent equations between the superfluous bar stresses S_{i_1}, S_{i_2}, etc. But of these latter there are N for each bar i, that is, the stress in the bar i under each one of the N different loadings, and therefore in all there are $k\,N$ of the superfluous bar stresses S_{i_1}, S_{i_2}, etc., between which the above $k\ (N-1)$ necessary relations subsist. It follows that the values of the S_i (and therefore the design itself), possess a k^{tuple} degree of indetermination, that is, in general, k of the S_i may be arbitrarily assumed, or the stress in each superfluous bar under one loading may be arbitrarily taken, or, more generally, any k further consistent relations between the stresses S_i may arbitrarily be set, and then a solution rigorously satisfying the given conditions of design as expressed through the primary equations, directly obtained. For, the solution of the equations under these circumstances furnishes us the stresses S_i in the superfluous bars under all the given loadings, thence at once all the stresses under the given loadings, thence through the prescribed conditions of design all the section areas, and finally thence, as previously explained, the precise unstrained lengths of the superfluous bars. But here a criticism is necessary. The complete mathematical solution of such a set of equations will, in general, consist of a considerable number of separate solutions, not all of which may be real, and of the real solutions, some or all of which may be inconsistent with conditions of design not expressible in the equations. These will generally be conditions of inequality such as that in the design thus obtained the most unfavorable demands shall be those supposed most unfavorable in drawing up the equations, and therefore the basis of those equations. For example, the rule of design being that each bar under its most unfavorable loading shall have stress of a prescribed intensity, in drawing up the equations, the most unfavorable loading for each bar must, in advance, be judged, the equations drawn up accordingly, and then the solutions thus obtained, which are real (and which will necessarily satisfy the condition that under the selected loading for each bar it has the prescribed intensity of stress), must further be tested as to their satisfying the suppositions as to most unfavorable loadings under which the equa-

* If two extremes of temperature were also to be considered, the effect would be to double the number of loadings. Temperature may be taken into consideration by putting $\frac{S_f}{E_f A_f} + \tau_f t_f$ for $\frac{S_f}{E_f A_f}$ where τ_f is the coefficient of expansion.

tions were drawn up. Frequently, none of the real mathematical solutions, where these exist, will satisfy these conditions, and thus limitations arise as to the arbitrary setting of stresses in the superfluous bars, or more generally, the arbitrary setting of k additional relations between these stresses. Nevertheless, a k^{tuple} infinity of designs of an indeterminate framework of given figure, under given loadings and subject to given conditions of design, is in general possible. It remains for us first to note some of the consequences, and second, to illustrate them by simple examples.

One will be most impressed, perhaps, by the fact of the possible multiplicity of exact and admissible designs of frameworks of given figure subject to given loadings. Clearly, any one of these designs will not necessarily be a good design, and, therefore, the fact that a design for an indeterminate framework of given figure subject to given loadings, rigorously fulfills the prescribed conditions of design is no indication whatsoever that the design in question is to be approved. Some such designs will be far more economic than others, that is, will require far less material; some such designs will be far stiffer than others, or will have other points of superiority. Our end is not simply to obtain any such design, but the best of such designs.

But what is the best such design? Suppose we say that design which is most economic of material is the best. The material in the bars is expressed by

$$\sum_{f=}^{f=m} A_f l_f$$

which is a function, through the prescribed conditions of design, of the bar stresses S_f, that is, finally, of the superfluous bar stresses S_t. We may determine mathematically the necessary relations for this to be a minimum, make these the additional arbitrary relations and thus obtain the most economic solutions. The most economic of these economic designs will be that sought. Theoretically the way is clear. And what will be the nature of this most economic design? No longer, as in the case of design under a single loading, will it necessarily be some statically determined framework of included figure. The fact that the volume of material no longer is expressible on any basis of design as a linear function of independent variables would alone

suggest this, and illustrations may easily be found which conclusively prove it. But, as will later be shown, this possible superiority of an indeterminate design, which will only occur when a structure is subject to very diverse principal loadings, and not at all, or in but small degree, to large structures, the dead weight of which is prominent, is never likely to be considerable, and only exists at all through the neglect of temperature stresses to which the indeterminate structure is subject where the determinate structure is not, and the further neglect of all consequences of dimensioning and work in erection of other than the most extreme accuracy, the neglect of any lack of absolute resistance of all supports, and the neglect of any yielding at the joints and set or other yielding under operation; all of which must in practice exist, and considerably affect the indeterminate construction where they inappreciably affect the determinate. Finally, simply as an interesting although perhaps not practically important fact, it may be observed that by a combination of statically determined designs, of figures included within that of a given indeterminate framework, or what the writer calls a multiple design, a series of different loadings can always be carried more economically with determinate stresses than by even the most economic of single indeterminate designs of the given figure, so that the conclusion is absolutely warranted that:

"Statical indetermination in a structure is always to be regarded as self-interference with efficiency."

Turning to the question of stiffness, but little can be added to what has already been said. Indeterminate designs subject to several loadings will in general have deflections of a different character from those of any one of the determinate designs of included figure. It is, therefore, impossible to set any basis of comparison that would not at once be subject to much criticism. The writer will merely observe that the comparisons between such structures, with which he is familiar, have usually been made on most unfair bases, and some too generally accepted conclusions, notably on the comparative stiffness of three-hinged and two-hinged or hingeless arches, are not justifiable. The writer's own very limited but fairly equable comparisons show clearly that no very marked superiority can be attributed to either type in this respect, and that points of superiority in one are offset by other

points of superiority in the other, so that nothing but a debatable conclusion in any case is possible. Some later illustrations will make this clear. And in fairness it must be conceded that the fact that in certain cases, in the attempt to secure static determination through the use of certain devices, play has been introduced, most objectionable in connection with loads moving at high speed, only shows the defective character of the devices used, and in no wise attains the principle the application of which does not necessarily involve the use of these devices.

Lastly, what of the superior safety of frameworks with superfluous bars? Since there are superfluous bars there are bars the rupture of which does not necessarily involve the failure of the framework, and this is certainly an element of safety; but, generally, not every bar may be regarded as a superfluous bar, so that in an indeterminate framework the rupture of some bars would not less necessarily involve the failure of the structure than in the case of a determinate framework. Finally, even of the bars the omission of which would leave the remaining portion statically determined, and, therefore, which, from this point of view, are not essential, some, perhaps many, will be of such actual importance in the structure that their removal would result in serious overstrain of some of the remaining bars, and thereby failure; while of those still remaining the sudden rupture of some and the accompanying shock would be fatal to the structure. The superior safety due to the use of superfluous bars is evidently very limited; and, when we note that the statically determined structures require less material to satisfy the same conditions of design, it becomes questionable whether for equal amounts of material the determinate design, which would then have lower unit stress than the indeterminate design, would not, therefore, be at least as safe.

Turning from these general considerations let us examine a little more closely the equations through the solution of which exact designs are to be obtained. The implicit and symbolic form in which we have stated them gives them a most deceptive appearance of simplicity which is very far from corresponding with facts. In explicit form we should find each equation to consist of a series of terms in the form of fractions, each fraction having for numerator a linear function of the variables S_i, but for denominator a linear or

higher degree function of the variables, according to the complexity of the rules of design used, and, very likely, it would not be a whole integral function. The simplest possible rule of design, making the section areas simply proportional to the greatest stress borne—which is the common rule for tension members—makes the denominator linear in terms of the variables S_i. But the introduction of post formulas of the Gordon-Rankine type, now generally used, at once would introduce fractional functions of the variable into the denominator. Now, these equations in fractions have a most delusive appearance of simplicity. Only when we have accomplished their reduction to whole integral equations, a very considerable although theoretically simple undertaking in itself, are we able to form some conception of what is involved in obtaining a solution. Then we discover that we have a series of simultaneous equations of higher degree than the first between numerous unknowns. The indetermination having been removed by the introduction of suitable arbitrary relations, by perfectly well-known methods of elimination, these equations are eventually reducible to a single equation with one unknown, the solution of which then by numerical methods (for, of course, we are dealing with equations with known numerical coefficients), offers no theoretic difficulty; but practically in any actual case the work involved would be appalling. And then the solutions obtained may prove worthless, as they may not be consistent with the suppositions on which the equations were drawn up. The number of such possible suppositions will generally be very considerable, and it may well happen that many trials will be necessary before admissible solutions will be found, thus multiplying the labor many times; and, if we wished to be complete in our mathematical solution, all the possible suppositions should be examined, making an undertaking from which even the bravest calculator might well recoil.

In order to give a conception, even though faint, of the appearance of one of the primary equations when written explicitly, but simply and in fractional form, such an equation for the bar i, when there are only the two different loadings 1 and 2, and when the basis of design is simply $A = \frac{S}{\sigma}$, is given, as follows:

$$\sum_{b=1}^{b=c} \frac{\sigma_b (S_{b_{0_1}} - S_{b_{0_2}})\, S'_b{}^{(i)} l_b}{E_b (S_{b_{0_1}} + \sum_{i=h}^{i=m} S_{i_1} S'_b{}^{(i)})} + \sum_{e=d}^{e=g} \frac{\sigma_e (S_{e_{0_1}} - S_{e_{0_2}})\, S'_e{}^{(i)} l_e}{E_e (S_{e_{0_2}} + \sum_{i=h}^{i=m} S_{i_2} S'_e{}^{(i)})}$$

$$+ (S_{h_1} - S_{h_2}) \left[\sum_{b=1}^{b=c} \frac{\sigma_b S'_b{}^{(h)} S'_b{}^{(i)} l_b}{E_b (S_{b_{0_1}} + \sum_{i=h}^{i=m} S_{i_1} S'_b{}^{(i)})} + \sum_{e=d}^{e=g} \frac{\sigma_e S'_e{}^{(h)} S'_e{}^{(i)} l_e}{E_e (S_{e_{0_2}} + \sum_{i=h}^{i=m} S_{i_2} S'_e{}^{(i)})} \right]$$

$$+ (S_{(h+1)_1} - S_{(h+1)_2}) \left[\sum_{b=1}^{b=c} \frac{\sigma_b S'_b{}^{(h+1)} S'_b{}^{(i)} l_b}{E_b (S_{b_{0_1}} + \sum_{i=h}^{i=m} S_{i_1} S'_b{}^{(i)})} + \sum_{e=d}^{e=g} \frac{\sigma_e S'_e{}^{(h+1)} S'_e{}^{(i)} l_e)}{E_e (S_{e_{0_2}} + \sum_{i=h}^{i=m} S_{i_2} S'_e{}^{(i)})} \right]$$

$$+ \cdots\cdots\cdots\

cases for investigation would have been further and greatly increased. So it will be seen that a complete, exact, mathematical solution, even on the simplest possible conditions of design, involves great labor, very rapidly increasing as the number of bars in the framework increases. The justness of the statement, made earlier, that the exact method was not to be thought of for use in practical designing, will now be perceived, and it is thought that the illustrations following will amply confirm this. But since exact methods are not applicable in practice, and the tentative methods of the past will continue to be used, and at best can never give better than most imperfect results, the impossibility of benefiting through the use of indeterminate designs, of even their theoretically possible advantages, will be clear; a final and fatal fact for the adherents to such construction. If ever an ideal has been completely vindicated by increased knowledge it is certainly that of the statically determined construction of frameworks.

Illustrations of Design of Indeterminate Frameworks under Multiple Loadings.

Let it be required to design a framework of the form and dimensions shown in Fig. A, so that it shall support either

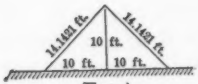


Fig. A.

the pull P in a horizontal direction, thus Fig. B. or the weight W, thus Fig. C. so that each bar, under its most unfavorable load, shall have stress of the mean intensity of 5 tons per square inch. Denoting the diagonals by the numbers 1 and 2, respectively, and the vertical Fig. D. by the number 3 (see Fig. D), and calling the loading with the horizontal pull the loading P, and the loading with the vertical weight the loading W, let us suppose that in the bars 1 and 2 the maximum stress occurs under the loading P and in the bar 3 under the loading W, so that

$$A_1 = \frac{S_{1_P}}{+5} \qquad A_2 = \frac{S_{2_P}}{-5} \qquad A_3 = \frac{S_{3_W}}{-5}$$

the bar 1 being designed under tensile stress and the bars 2 and 3 under compression.

Treat bar 3 as the superfluous bar. Then unity tension in it would cause compression of 0.707 in each of the bars 1 and 2 or $S'_1 = S'_2 = -0.707$ (see Fig. E). The apex of the triangle sinks more under the load W than under the pull P by

FIG. E.

$$\frac{S'_1 (S_{1_W} - S_{1_P}) l_1}{EA_1} + \frac{S'_2 (S_{2_W} - S_{2_P}) l_2}{EA_2} = (S_{3_P} - S_{3_W}) \frac{l_3}{EA_3}$$

and, noting the values of S', l and A, this gives the simple relation

$$\frac{S_{1_P} - S_{1_W}}{S_{1_P}} + \frac{S_{2_P} - S_{2_W}}{-S_{2_P}} = \frac{S_{3_P} - S_{3_W}}{-S_{3_W}}$$

$$\text{Now, } S_{1_P} = S_{1_{o_P}} + S'_1 S_{3_P} = +0.707P - 0.707S_{3_P}$$

$$S_{1_W} = S_{1_{o_W}} + S'_1 S_{3_W} = -0.707W - 0.707S_{3_W}$$

$$S_{2_P} = S_{2_{o_P}} + S'_2 S_{3_P} = -0.707P - 0.707S_{3_P}$$

$$S_{2_W} = S_{2_{o_W}} + S'_2 S_{3_W} = -0.707W - 0.707S_{3_W}$$

so that the relation may be written

$$\frac{P + W - S_{3_P} + S_{3_W}}{P - S_{3_P}} - \frac{P - W + S_{3_P} - S_{3_W}}{P + S_{3_P}} = \frac{S_{3_W} - S_{3_P}}{S_{3_W}}$$

or, clearing of fractions,

$$2PS^2_{3_W} + (2PW - P^2 + S^2_{3_P}) S_{3_W} + P^2 S_{3_P} - S^3_{3_P} = 0$$

which is a quadratic equation in S_{3_W} and a cubic in S_{3_P}. We observe the following relations:

If $S_{3_W} = 0$, then $S_{3_P} = 0$, the bar 3 having zero area.

If $S_{3_P} = \pm P$, then $S_{3_W} = -W$, either bar 1 or bar 2 having zero area.

If $S_{3_P} = 0$, then $S_{3_W} = -W + \frac{P}{2}$, bars 1 and 2 being of the same area and bearing together $\frac{P}{2}$, whereby their junction sinks the full allowable strain of 3.

We note that S_{3_P} is the primary stress in bar 3, since it does not vary with P, and the primary stresses in 1 and 2 are $-0.707 S_{3_P}$.

It will be seen that the bar 3 can only be used to advantage when W is greater than P, which we will suppose to be the case.

Since 1 and 2 are most efficient in resisting P we have supposed them designed under P, while 3, which is only efficient in resisting W,

we have supposed designed under W. Thus we must satisfy the inequalities

$$S^2_{1P} \geqq S^2_{1W} \qquad S^2_{1P} \geqq \frac{S^2_{3P}}{2}$$

$$S^2_{3W} \geqq S^2_{3P} \text{ and also}$$

$$S^2_{2P} \geqq S^2_{2W} \qquad S^2_{2P} \geqq \frac{S^2_{3P}}{2}$$

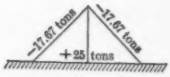
The latter pair of inequalities are necessitated by the primary stresses.

A consequence of this last pair of inequalities is that $S^2_{3P} \leqq \left(\frac{P}{2}\right)^2$ so that $\frac{(S_{1P} - S_{2P})\, l_1}{5}$ is the volume of material in the bars 1 and 2 under these circumstances, and is constant. For economy then, we must so choose S_{3P} within the above limits that, while satisfying the remaining inequalities, S^2_{3W} shall be as small as possible.

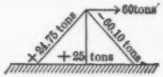
Let us take $W = 100$ tons, $P = 60$ tons, and assume $S_{3P} = 25$ tons. We have $S^2_{3W} + \frac{9025}{120} S_{3W} + (750 - \frac{15625}{120}) = 0$, whence $S_{3W} = -37.6 \pm 28.2 = -9.4$ tons, or -65.8 tons, of which the former is excluded by the condition $S^2_{3W} \geqq S^2_{3P}$.

Taking the solution of $S_{3W} = -65.8$ tons, we obtain the following:

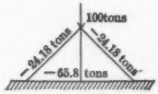
With no load, $S_1 = S_2 = -17.67$ tons, $S_3 = +25$ tons.



With load $P = 60$ tons, $S_{1P} = +24.75$ tons, $S_{2P} = -60.10$ tons, $S_{3P} = +25$ tons.



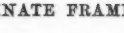
With load $W = 100$ tons, $S_{1W} = -24.18$ tons, $S_{2W} = -24.18$ tons, $S_{3W} = -65.8$ tons.



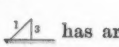
The section areas are: (bar 1) $\frac{24.75}{5} = 4.95$ sq. ins., (bar 2) $\frac{-60.10}{-5} = 12.02$ sq. ins., and (bar 3) $\frac{-65.8}{-5} = 13.16$ sq. ins.

The volumes are: (bar 1) 840.0 cu. ins., (bar 2) 2 039.8 cu. ins., (bar 3) 1 579.2 cu. ins., a total of 4 459 cu. ins.

Consider the following comparisons:

The statically determined form  has areas (bar 1) 16.97 sq. ins., (bar 3) 20 sq. ins.; and volumes (bar 1) 2 879.8 cu. ins., (bar 3) 2 400 cu. ins., a total of 5 279.8 cu. ins.

The statically determined form has areas (bars 1 and 2) each 14.14 sq. ins.; and volumes (bars 1 and 2) each 2 400 cu. ins., a total of 4 800 cu. ins.

But the combination or "multiple" form for P and  for $W - P$ has areas (bars 1 and 2) each 8.48 sq. ins., and (bar 3) 8 sq. ins.; and volumes (bars 1 and 2) each 1 440 cu. ins., (bar 3) 960 cu. ins., a total of but 3 840 cu. ins.

This illustrates clearly how the indeterminate form may be more economic than any single included determinate form, yet not as economic as a combination of determinate forms.

The second value of S (— 9.4 tons), excluded by the condition $S^2_{3P} \geqq S^2_{3W}$, appears in the solution correctly as one of the answers to the unrestricted problem. This partial exclusion of solutions, as previously noted, would usually occur in problems of this description, and often not even one of the solutions would be admissible under the limitations prescribed.

Thus, in this case, S_{1P} being tension and S_{1W} being compression, the condition $S^2_{1P} \geqq S^2_{1W}$ may be written $S_{1P} \geqq -S_{1W}$; whence $P - S_{3P} \gtrless W + S_{3W}$. The limiting value of this relation is $P - W - S_{3P} = S_{3W}$, or, in this case, $40 + S_{3P} = -S_{3W}$. If we substitute this in the equation

$$\frac{P + W - S_{3P} + S_{3W}}{P - S_{3P}} - \frac{P - W + S_{3P} - S_{3W}}{P + S_{3P}} + \frac{S_{3P} - S_{3W}}{S_{3W}} = 0$$

this becomes $2 - \frac{2S_{3P}}{60 + S_{3P}} - \frac{40 + 2S_{3P}}{40 + S_{3P}} = 0$; or $S^2_{3P} + 20S_{3P} - 1\,200 = 0$, a quadratic, of which $S_{3P} = +26.056$ is one root, and the corresponding value of S_{3W} is $-(40 + S_{3P}) = -66.05$. If we were to assume S_{3P} greater than $+26.056$, there would be no admissible solution.

At first glance it would seem that the greater S_{3P}, the smaller (numerically) would be S_{3W}; but if we differentiate the equation of

relation between S_{3P} and S_{3W} and put $\frac{dS_{3W}}{dS_{3P}} = 0$, we obtain the condition

$$S_{3W} = \frac{3}{2}\left(\frac{S_{3P}^2 - \frac{1}{3}P^2}{S_{3P}}\right) \text{ or, in this case, } S_{3W} = \frac{3}{2}\left(\frac{S_{3P}^2 - 1\,200}{S_{3P}}\right),$$

and substituting this in the equation of relation between S_{3P} and S_{3W}, we obtain the quintic

$$S_{3P}^6 + 540 S_{3P}^4 + 28\,800 S_{3P}^3 - 1\,296\,000 S_{3P}^2 - 30\,240\,000 S_{3P} + 777\,600\,000 = 0.$$

One root is $S_{3P} = +19.168$, and the corresponding (minimum) value of S_{3W} is $S_{3W} = -65.14$, which gives only slightly more economy than $S_{3P} = +25$.

Thus far we have examined only the case of bars 1 and 2 with maximum loads under P, and bar 3 with maximum load under W, and, moreover, the suppositions $S_{1P} > 0$, $S_{2P} < 0$, $S_{3W} < 0$. These latter were not, perhaps, necessary, but apparently are essential to economic solutions. On the other hand, although the argument that the bars 1 and 2 should have their maximum loads under P for a most economic design looks reasonable, it may be misleading, so let us examine some further cases.

As a second case suppose bar 1 to have its maximum stress under W (instead of under P), bar 2 having its maximum stress under P, and bar 3 its maximum stress under W as before. Here we will suppose $S_{1W} < 0$, $S_{2P} < 0$ and $S_{3W} < 0$, so that the bars are all designed under compression. The equation is

$$\frac{S_{1P} - S_{1W}}{-S_{1W}} + \frac{S_{2P} - S_{2W}}{-S_{2P}} = \frac{S_{3P} - S_{3W}}{-S_{3W}}$$

or, replacing S_{1P}, S_{1W}, S_{2P} and S_{2W} by their values in P, W, S_{3P} and S_{3W},

$$\frac{P + W - S_{3P} + S_{3W}}{W + S_{3W}} - \frac{P - W + S_{3P} - S_{3W}}{P + S_{3P}} + \frac{S_{3P} - S_{3W}}{S_{3W}} = 0,$$

which, cleared of fractions, gives

$$W S_{3P}^2 + (P\,W + P\,S_{3W} - W\,S_{3W} - S_{3W}^2)\,S_{3P} + (P^2 - P\,W + W^2)\,S_{3W} - (P - 2\,W)\,S_{3W}^2 + S_{3W}^3 = 0,$$

which is a quadratic in S_{3P} and a cubic in S_{3W}. Now, we have in this case

$$A_1 = \frac{S_{1W}}{-5} = \frac{0.707\,(W + S_{3W})}{5} \text{ sq. ins.} \qquad l_1 = 12 \times 14.14 \text{ ins.}$$

$$A_2 = \frac{S_{2P}}{-5} = \frac{0.707\,(P + S_{3P})}{5} \text{ sq. ins.} \qquad l_2 = 12 \times 14.14 \text{ ins.}$$

$$A_3 = \frac{S_{3W}}{-5} = \frac{-S_{3W}}{5} \text{ sq. ins.} \qquad l_3 = 12 \times 10 \text{ ins.}$$

$$\text{Total volume} = A_1 l_1 + A_2 l_2 + A_3 l_3 = 24\,(P + W + S_{3P}).$$

It is clear that for economy we must aim to make S_{3P} as small as possible while satisfying the inequalities $S^2_{1W} > S^2_{1P}$, $S^2_{2P} > S^2_{2W}$, $S^2_{3W} > S^2_{3P}$. The first two of these conditions combined require $S^2_{2P} > S^2_{1P}$, since $S_{1W} = S_{2W}$ necessarily; and thence follows that $S_{3P} > 0$. Since $S_{2P} \leqq 0, -S_{2P} \geqq -S_{2W}$, or $P - W + S_{3P} - S_{3W} \geqq 0$. If $P + S_{3P} = W + S_{3W}$, then $\dfrac{P + W - S_{3P} + S_{3W}}{W + S_{3W}} + \dfrac{S_{3P} - S_{3W}}{S_{3W}} = 0$;

whence $S_{3P} = P\,\dfrac{W - P}{W + P} = +15$ tons, $S_{3W} = -25$ tons.

Now, $S_{3P} - S_{3W}$ diminishes with S_{3P} at these values, and they alone satisfy $P + S_{3P} - W - S_{3W} = 0$. Thus, if $P + S_{3P} - W - S_3 \geqq 0$, then $S_{3P} \geqq P\,\dfrac{W - P}{W + P}$. At the limit we may take $S_{3P} = P\,\dfrac{W - P}{W + P}$ $= 15$ tons, and this is the least allowable value of S_{3P} in this case. The corresponding values are:

$S_{3W} = P - W + S_{3P} = -25$ tons,

$S_{2P} = S_{2W} = S_{1W} = -0.707\,(P + S_{3P}) = -53.02$ tons,

$S_{1P} = 0.707\,(P - S_{3P}) = +31.81$ tons,

$A_1 = A_2 = 10.60$ sq. ins.,

$A_3 = 5$ sq. ins.,

Vol. bar 1 = vol. bar 2 = 1 800 cu. ins.,

Vol. bar 3 = 600 cu. ins.

Total volume of bars = 4 200 cu. ins., which is a considerably more economic design than that of our first case. We note that, since $A_1 = A_2$ $S_{3P} = +15$ tons is also the primary stress in bar 3.

As a third case, suppose bar 1 has its maximum stress under P, but bar 2 has its maximum stress under W, and bar 3 has its maximum stress under W as before. In this case $S^2_{1P} > S^2_{1W}$, $S^2_{2W} > S^2_{2P}$, $S^2_{3W} > S^2_{3P}$.

Since $S_{1W} = S_{2W}, S^2_{1P} > S^2_{2P}$; whence $S_{3P} < 0$, $S_{1P} > 0$ and $S_{3W} < 0$. This latter follows from $S_{1P} > -S_{1W}$, or $P - S_{3P} > W + S_{3W}$ and $S^2_{3W} > S^2_{3P}$.

Now, if we suppose $S_{2W} = S_{1W} < 0$, then the equation would be

$$\frac{S_{1P} - S_{1W}}{S_{1P}} + \frac{S_{2P} - S_{2W}}{-S_{2W}} = \frac{S_{3P} - S_{3W}}{-S_{3W}}$$

which we may write $1 - \frac{S_{1W}}{S_{1P}} - \frac{S_{2P}}{S_{2W}} + \frac{S_{3P}}{S_{3W}} = 0$. Since the terms following the first are numerically less than unity by supposition, this equation asserts that $1 + (>0) \pm (<1) + (>0) = 0$, which is impossible.

If we suppose $S_{2W} = S_{1W} > 0$, then the equation would be

$$\frac{S_{1P} - S_{1W}}{S_{1P}} + \frac{S_{2P} - S_{2W}}{S_{2W}} = \frac{S_{3P} - S_{3W}}{-S_{3W}}$$

which we may write $-1 - \frac{S_{1W}}{S_{1P}} + \frac{S_{2P}}{S_{2W}} + \frac{S_{3P}}{S_{3W}} = 0$

or $-1 - (>0) \pm \left(\begin{matrix}<1\\>0\end{matrix}\right) + (<1) = 0$, which is possible only if $S_{2P} > 0$, that is, if $P + S_3 < 0$, or if $S_{3P} < -P$. Since $S_{2W} > 0$, $-(W + S_{3W}) > 0$, or $S_{3W} < -W$. Now, $A_1 = \frac{0.707\,(P - S_{3P})}{5}$, $A_2 = \frac{-0.707\,(W + S_{3W})}{5}$, $A_3 = \frac{-S_{3W}}{5}$, $A_1 l_1 + A_2 l_2 + A_3 l_3 = 24\,(P - W - S_{3P} - 2\,S_{3W})$ cu. ins. This will be a minimum for $S_{3P} + 2\,S_{3W}$ a maximum. But the maximum values $S_{3P} = -P$ and $S_{3W} = -W$ occur simultaneously, making $S_{3P} + 2\,S_{3W} = -(P + 2\,W)$ a maximum, therefore, we have

$S_{3P} = -60$ tons,	$A_1 = 16.96$ sq. ins.,	Vol. bar $1 = 2\,880$ cu. ins.,
$S_{3W} = -100$ tons,	$A_2 = 0$,	Vol bar $2 = 0$,
$S_{1P} = +84.84$ tons,	$A_3 = 20$ sq. ins.,	Vol. bar $3 = 2\,400$ cu. ins.
$S_{2P} = S_{1W} = S_{2W} = 0$,		

Total volume, $= 24\,(60 - 100 + 60 + 200) = 5\,280$ cu. ins.

This "economic" solution is a statically determined form.

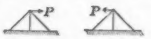
As a fourth case let all the bars have their maximum stresses under W, then clearly $S_{1W} < 0$, $S_{2W} < 0$, and the equation is

$$\frac{S_{1P} - S_{1W}}{-S_{1W}} + \frac{S_{2P} - S_{2W}}{-S_{2W}} = \frac{S_{3P} - S_{3W}}{-S_{3W}}$$

Here $A_1 = A_2$, and consequently S_{3P} is the primary stress in 3. $A_1 = \frac{S_{1W}}{-5} = \frac{0.707\,(W + S_{3W})}{5} = A_2$, $A_3 = \frac{-S_{3W}}{5}$, $A_1 l_1 + A_2 l_2 + A_3 l_3 = 24\,(2\,W + S_{3W})$ cu. ins. This will be a minimum for S_{3W} a minimum. Now, from $S^2_{2W} > S^2_{2P}$ follows $W + S_{3W} > P + S_{3P}$, or $S_{3W} > P - W + S_{3P}$, or $W + S_{3W} - S_{3P} > P$. But $\frac{S_{1P} - S_{1W}}{-S_{1W}} + \frac{S_{2P} - S_{2W}}{-S_{2W}} + \frac{S_{3P} - S_{3W}}{+S_{3W}} = 0$ gives $(W + S_{3W} - S_{3P})\,S_{3W} + W\,S_{3P} = 0$; whence $P\,S_{3W} + W\,S_{3P} > 0$, since $S_{3W} < 0$. This gives $S_{3P} > -\frac{P}{W} S_{3W}$; whence $S_{3W} > P - W - \frac{P}{W} S_{3W}$, or $S_{3W} > W\,\frac{P - W}{P + W}$, or $S_{3W} > -25$. At limit $S_{3W} = -25$ tons (minimum), $S_{3P} = +15$ tons, $S_{1P} = +31.81$ tons, $S_{2P} = S_{2W} = S_{1W} = -53.02$ tons; $A_1 = A_2 = 10.60$ sq. ins., $A_3 = 5$ sq. ins. Total volume $= 24\,(200 - 25) = 4\,200$ cu. ins., which is identical with the most economic design of the second case.

We might consider further cases in which bar 3 was designed under the load P, but it seems sufficiently evident that these could not be economic designs and need no study.

The final conclusion is that in the most economic design all bars have their maximum stresses under the vertical load, and one of the bars has the same stress under the horizontal pull, a result somewhat different from that anticipated at the outset.

As a last study, suppose it were required that the frame should support any one of the three loadings 

, then solutions of the first three of these preceding cases would not in general apply, since they are non-symmetrical. But solutions of the last of the cases preceding evidently would apply, since they are symmetrical, and the most economic solution will be the same here as there. That is to say, the design with $A_1 = A_2 = 10.60$ sq. ins., $A_3 = 5$ sq. ins., and having a volume of 4 200 cu. ins., and a primary stress in the vertical of $+15$ tons, is the most economic design for the three loadings as shown. This is as economic as the best design for the smaller number of loadings, but usually the economic design for the smaller number of loadings will be the more economical, and, evidently, it will never be the less economical.

This example, perhaps the simplest that could be selected in the design of indeterminate frameworks subject to multiple loadings, will furnish a conception of the difficulty, or rather the practical impossibility, of any general application of such analysis. Here, for single solutions independent of limiting conditions, the solution of quadratic equations was necessary, in one of the cases the solution of a quintic was involved in the determination of economic proportions, and last, but perhaps not least, the examination of numerous cases of defining loadings together with puzzling questions in connection with limiting conditions, had to be gone through with before the most economic design was found and established as such. In any other case the work of this case (which is hardly more than outlined here), and the difficulties would be multiplied many fold. The equations would be of much higher degree and simultaneous, the different cases of defining loadings to be examined would be far more numerous, and the consequences of the limiting conditions would be infinitely more complex and difficult to trace out. In problems of a practical description the quantity and complexity of the work would be such that it is safe to regard it as prohibitive.

Illustration of the Design of an Indeterminate Arch Under Several Loadings.

As a second example consider the two-hinged arch shown in Fig. 1.

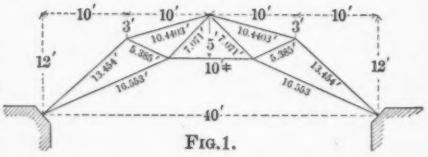


Fig. 1.

The dimensions of the arch are as follows: Span, 40 ft., divided into four parts of 10 ft. each by the upper chord, the rise being 12 ft., and the quarter points being on the parabola through the vertex and the supports. The middle tie is 10 ft. long and 5 ft. from the vertex and all other dimensions follow as shown.

The arch being symmetrical, it is most desirable to denote corresponding bars in a clear manner, which has been done by giving each symmetric pair the same letter, and following it with l or r in subscript to distinguish the bars on the left from those on the right.

The middle bar, which alone has no duplicate, has been denoted by the single letter z (see Fig. 2).

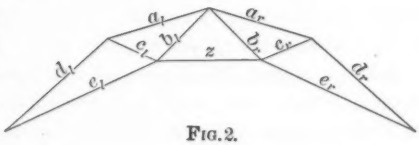


FIG. 2.

The bar z we will treat as the superfluous bar. The stresses S'_{a_r}, S'_{b_r}, etc., due to a tension of unity in z, are as shown in Fig. 3.

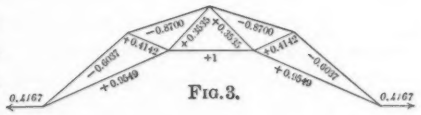
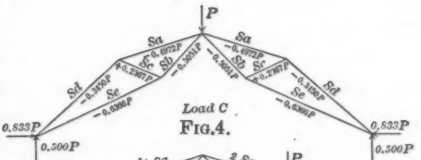


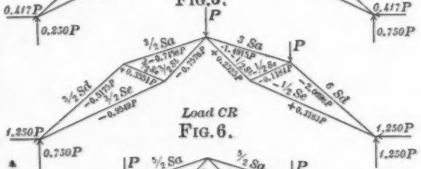
FIG. 3.

We will suppose only the three upper chord joints to be points of loading, neglect dead weight, and consider the various loadings formed by applying the same vertical load P to any one or more of the three points. Denoting the single loadings by L, C or R, according as the load is on the left, center or right joint, we have in single and combined loadings the eight cases, L, C, R, LC, CR, LR, LCR, and "no loading," but of these L and R and LC and CR form symmetrical cases, reducing the loadings requiring separate consideration to six if we confine ourselves to symmetric designs. And only two of these loadings, in addition to the case of "no

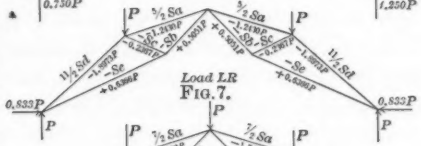


Load C
FIG. 4.

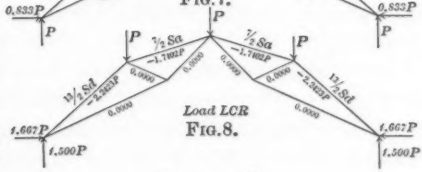
Load R
FIG. 5.



Load CR
FIG. 6.



Load LR
FIG. 7.



Load LCR
FIG. 8.

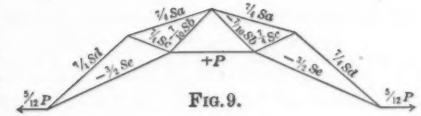


FIG. 9.

loading," require separate calculation, since the others can be obtained through suitably combining these.

The stresses under these various loadings occurring in the statically determined form, that is to say, the three-hinged arch remaining when the middle bar z is omitted, are shown in Figs. 4, 5, 6, 7 and 8.

It will be observed that the stresses of Figs. 6, 7 and 8 are obtained from Figs. 4 and 5 simply by addition.

Denoting the stresses under load C (Fig. 4) by S_a, S_b, S_c, S_d and S_e simply, we find that the stresses under load R (Fig. 5) may be very conveniently expressed in terms of these, and similarly for all the other loadings (Figs. 6, 7 and 8).

Moreover, comparing Fig. 3 with Fig. 4, we also note relations such that if the tension in z were $+P$, then the stresses, in terms of those in Fig. 4, would be as shown in Fig. 9. Therefore, for convenience in the following we will express all stresses in terms of P, which may be omitted from the equations, but which must be taken into consideration when we come to numerical computations.

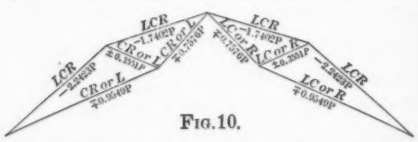


FIG. 10.

Bar	a	b	c	d	e	Total
Volume	$18.166\frac{P}{\sigma}$	$5.357\frac{P}{\sigma}$	$1.912\frac{P}{\sigma}$	$30.174\frac{P}{\sigma}$	$15.806\frac{P}{\sigma}$	$142.830\frac{P}{\sigma}$

Fig. 10 shows the maximum bar stresses of the three-hinged arch together with the loads under which they occur. The bar volumes and total which would correspond to this as a design are also given. We have to find what economy over this can be secured through introducing the middle bar z and making the design indeterminate.

In order to enable us to form a conception of the changes in stress in the indeterminate design, it will be of service to know the changes δz in the middle bar distance under the various loadings in the case of this determinate three-hinged design; for the indeterminate design certainly will have somewhat similar proportions. The shortening of the distance z is $\delta z = \Sigma \frac{S' Sl}{EA}$. Now the areas are the maximum stresses as given in Fig. 10 divided by the working stress; the quotients being taken posi-

tive. Or we may write $\frac{E}{\sigma}\,\delta\, z = \Sigma\, S\, \frac{S'\, l}{S_{max}}$. In Fig. 11 are the values of $\frac{S'\, l}{S_{max}}$ with the signs of the S', and by aid of these the values of $\frac{E}{\sigma}\,\delta\, z$ were calculated for all the loadings as shown. The range of $\frac{E}{\sigma}\,\delta\, z$ exceeds 60. When the bar z is introduced, that must be so resistent as to reduce the range of $\frac{E}{\sigma}\,\delta\, z$ within 20 since z being 10 ft. long, δz may not vary more than $10\,\frac{\sigma}{E}$ either way from its mean.

Now if we consider the stresses as shown in Figs. 4 to 8, and their modification through introducing a bar z, by aid of the stresses shown in Fig. 3 or Fig. 9, and in the light of this last data as to the character of the variations $\delta\, z$, we see readily that for greatest economy of material in the bars b, c and e the stress in z under L or R must be equal but opposite in sign to that under LC or CR, and similarly under LR equal but of opposite sign to that under C. Therefore the primary stress in z should be equal but of opposite sign to that in z under the full load LCR. This requires that with $\frac{1}{2}\,P$ on each point the bar z shall be free from stress. The greatest stress in z, and therefore the material used in z in making its greatest strain but $\frac{E}{\sigma}$, would also be least for this condition of design, so that it seems probable that this will furnish a, if not the, most economic, indeterminate design. Only the volumes of the bars a and d would be favored by further increasing the resistance of z under the loading LCR, and it does not seem probable that gains there would offset the consequent losses by the bars b, c, e and z. We will therefore seek the particular solution for which the stress in z would be zero with $\frac{1}{2}\,P$ on each point, therefore, for which the stress in z under the various loadings gives the following relations: $S_{z_R} = S_{z_L} = -S_{z_{CR}} = -S_{z_{LC}} = S_5$ say, (Fig. 5 corresponding to loading R), $S_{z_{LR}} = -S_{z_C} = S_7$ say, and $S_{z_{LCR}} = -S_{z_0}$ (the primary stress) $= S_8$ say.

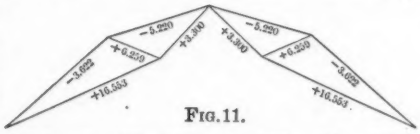


FIG. 11.

Load	C	CR or LR	R or L	LCR	LR	$S_z = P$
$\frac{E}{\sigma}\delta\, z$	−13.756	+10.328	+24.084	+34.412	+48.168	+52.587

We are now confronted with the question: Under what loadings

will the bars probably have their maximum stresses? It would seem likely that, as in the three-hinged arch, a and d have their greatest stresses under LCR; b, c and e under R and L (CR and LC); and z, judging by the data of Fig. 11, under LR (and C). We will, therefore, first proceed on this basis. This gives the expressions for the section areas A, and the Figs. 4 to 9 enable us to write the stresses as follows:

$$A_a = \frac{-S_{a_{LCR}}}{\sigma} \qquad S_{a_o} = -\tfrac{7}{4}S_a S_8 \qquad S_{a_{LCR}} = +\tfrac{7}{4}S_a\,(2 + S_8)$$

$$A_b = \frac{S_{b_{r_R}}}{\sigma} \qquad S_{b_o} = +\tfrac{7}{10}S_b S_8 \qquad S_{b_{LCR}} = -\tfrac{7}{10}S_b S_8$$

$$A_c = \frac{-S_{c_{r_R}}}{\sigma} \qquad S_{c_o} = -\tfrac{7}{4}S_c S_8 \qquad S_{c_{LCR}} = +\tfrac{7}{4}S_c S_8$$

$$A_d = \frac{-S_{d_{LCR}}}{\sigma} \qquad S_{d_o} = -\tfrac{7}{4}S_d S_8 \qquad S_{d_{LCR}} = +\tfrac{1}{4}S_d(26 + 7S_8)$$

$$A_e = \frac{S_{e_{r_R}}}{\sigma} \qquad S_{e_o} = +\tfrac{3}{2}S_e S_8 \qquad S_{e_{LCR}} = -\tfrac{3}{2}S_e S_8$$

$$A_z = \frac{-S_{z_{LR}}}{\sigma} = \frac{-S_7}{\sigma}$$

$$S_{a_{LR}} = +\tfrac{1}{4}S_a(10 + 7S_7) \qquad S_{a_{r_R}} = +\tfrac{1}{4}S_a(8 + 7S_5)$$

$$S_{b_{LR}} = -\tfrac{1}{10}S_b(10 - 7S_7) \qquad S_{b_{r_R}} = -\tfrac{1}{10}S_b(15 + 7S_5)$$

$$S_{c_{LR}} = -\tfrac{1}{4}S_c(4 - 7S_7) \qquad S_{c_{r_R}} = -\tfrac{1}{4}S_c(6 - 7S_5)$$

$$S_{d_{LR}} = +\tfrac{1}{4}S_d(22 + 7S_7) \qquad S_{d_{r_R}} = +\tfrac{1}{4}S_d(20 + 7S_5)$$

$$S_{e_{LR}} = -\tfrac{1}{2}S_e(2 + 3S_7) \qquad S_{e_{r_R}} = -\tfrac{3}{2}S_e(1 + S_5)$$

$$S_{a_C} = +\tfrac{1}{4}S_a(4 - 7S_7) \qquad S_{a_{l_{CR}}} = +\tfrac{1}{4}S_a(6 - 7S_5)$$

$$S_{b_C} = +\tfrac{1}{10}S_b(10 + 7S_7) \qquad S_{b_{l_{CR}}} = +\tfrac{1}{10}S_b(15 + 7S_5)$$

$$S_{c_C} = +\tfrac{1}{4}S_c(4 - 7S_7) \qquad S_{c_{l_{CR}}} = +\tfrac{1}{4}S_c(6 - 7S_5)$$

$$S_{d_C} = +\tfrac{1}{4}S_d(4 - 7S_7) \qquad S_{d_{l_{CR}}} = +\tfrac{1}{4}S_d(6 - 7S_5)$$

$$S_{e_C} = +\tfrac{1}{2}S_e(2 + 3S_7) \qquad S_{e_{l_{CR}}} = +\tfrac{3}{2}S_e(1 + S_5)$$

$$S_{a_{l_R}} = +\tfrac{1}{4}S_a(2 + 7S_5) \qquad S_{a_{r_{CR}}} = +\tfrac{1}{4}S_a(12 - 7S_5)$$

$$S_{b_{l_R}} = +\tfrac{1}{10}S_b(5 - 7S_5) \qquad S_{b_{r_{CR}}} = -\tfrac{1}{10}S_b(5 - 7S_5)$$

$$S_{c_{l_R}} = +\tfrac{1}{4}S_c(2 + 7S_5) \qquad S_{c_{r_{CR}}} = -\tfrac{1}{4}S_c(2 + 7S_5)$$

$$S_{d_{l_R}} = +\tfrac{1}{4}S_d(2 + 7S_5) \qquad S_{d_{r_{CR}}} = +\tfrac{1}{4}S_d(24 - 7S_5)$$

$$S_{e_{l_R}} = +\tfrac{1}{2}S_e(1 - 3S_5) \qquad S_{e_{r_{CR}}} = -\tfrac{1}{2}S_e(1 - 3S_5)$$

We suppose the coefficient of elasticity to be the same for all bars, therefore, between any two loadings, as $L\,C\,R$ and 0 (no loading), we have equations such as $\Sigma \frac{S'(S_{LCR}-S_o)\,l}{A} = 0$. Writing these for the three pairs of symmetric loadings LCR and 0, R and $C\,R$, and $L\,R$ and C, we have—

$$\frac{-1}{2\sigma}\Sigma\frac{S'(S_{LCR}-S_o)\,l}{A} \equiv \frac{S_a(7+7S_8)\,l_a}{4+2S_8} + \frac{49S_bS_8l_b}{75+35S_5} + \frac{49S_cS_81_c}{-12+14S_5} + \frac{7S_d(13+7S_8)\,l_d}{52+14S_8} + \frac{3S_eS_8l_e}{1+S_5} + \frac{S_8l_z}{S_7} = 0$$

$$\frac{-1}{2\sigma}\Sigma\frac{S'(S_R-S_{CR})l}{A} \equiv \frac{S_a(-2+7S_5)\,l_a}{4+2S_8} + \frac{7S_b(5+7S_5)\,l_b}{75+35S_5} + \frac{7S_c(-2+7S_5)\,l_c}{-12+14S_5} + \frac{7S_d(-2+7S_5)\,l_d}{52+14S_8} + \frac{S_e(1+3S_5)\,l_e}{1+S_5} + \frac{S_5l_z}{S_7} = 0$$

$$\frac{-1}{2\sigma}\Sigma\frac{S'(S_{LR}-S_C)\,l}{A} \equiv \frac{S_a(3+7S_7)\,l_a}{4+2S_8} + \frac{7S_b(10+7S_7)\,l_b}{75+35S_5} + \frac{7S_c(-4+7S_7)\,l_c}{-12+14S_5} + \frac{7S_d(9+7S_7)\,l_d}{52+14S_8} + \frac{S_e(2+3S_7)\,l_e}{1+S_5} + l_z = 0$$

Now, if to the first of the above equations we add twice the second and then subtract the third, we obtain the simple relation $S_8 + 2S_5 - S_7 = 0$. This we might otherwise have perceived from the following reasoning: The loading $L\,R$ is obtained by adding to the loading $\frac{1}{2}P$, $\frac{1}{2}P$, $\frac{1}{2}P$ the loading $\frac{1}{2}P$,$-\frac{1}{2}P$, $\frac{1}{2}P$. The loading $L\,C\,R$ is obtained by adding to the loading $\frac{1}{2}P$, $\frac{1}{2}P$, $\frac{1}{2}P$, the loading $\frac{1}{2}P$, $\frac{1}{2}P$, $\frac{1}{2}P$. And half the difference of these additive loads (viz., $\frac{1}{2}\,[(\frac{1}{2}P,-\frac{1}{2}P,\frac{1}{2}P)-(\frac{1}{2}P, \frac{1}{2}P,\frac{1}{2}P)] = (0,-\frac{1}{2}P,0)$, added to $\frac{1}{2}P$, $\frac{1}{2}P$, $\frac{1}{2}P$ gives the loading $\frac{1}{2}P$, 0, $\frac{1}{2}P$ which would cause the same strain of z as the loading 0, 0, P or loading R. This is a check of the above equations.

Substituting in the second and third of these equations $S_8 + 2S_5$ for S_7 and clearing of fractions in S_8 we get

$$28\,\alpha S^3{}_8 + \begin{bmatrix} (56S_5+160)\,\alpha \\ +14S_a(-2+7S_5)l_a \\ +14S_d(-2+7S_5)l_d \\ +28S_5\,l_z \end{bmatrix} S^2{}_8 + \begin{bmatrix} (320S_5+208)\,\alpha \\ +(28S_5+52)\,S_a(-2+7S_5)l_a \\ +(4S_5+4)\,7S_d(-2+7S_5)l_d \\ +160S_5\,l_z \end{bmatrix} S_8 + \begin{bmatrix} 416S_5\,\alpha \\ +104S_5\,S_a(-2+7S_5)\,l_a \\ +56S_5\,S_d(-2+7S_5)\,l_d \\ +208S_5\,l_z \end{bmatrix} = 0$$

where $\alpha \equiv \frac{7S_b\,(5+7S_5)\,l_b}{75+35S_5} + \frac{7S_c\,(-2+7S_5)\,l_c}{-12+14S_5} + \frac{S_e\,(1+3S_5)\,l_e}{1+S_5}$,

and

$$28\,\beta S^3{}_8 + \begin{bmatrix} 160\beta \\ +\,28\gamma \\ +\,98S_a\,l_a \\ +\,98S_d\,l_d \end{bmatrix} S^2{}_8 + \begin{vmatrix} 208\beta \\ +\,160\gamma \\ +\,14S_a\,(3+14S_5)\,l_a \\ +\,14S_d\,(9+14S_5)\,l_d \\ +\,364S_a\,l_a \\ -\,+\,196S_d\,l_d \end{vmatrix} S_8$$

$$+ \begin{bmatrix} 208\gamma \\ +\,52S_a\,(3+14S_5)\,l_a \\ +\,28S_d\,(9+14S_5)\,l_d \end{bmatrix} = 0$$

where $\beta \equiv \frac{49S_b\,l_b}{75+35S_5} + \frac{49S_c\,l_c}{-12+14S_5} + \frac{3S_e\,l_e}{1+S_5}$.

$$\gamma \equiv \frac{7S_b\,(10+14S_5)\,l_b}{75+35S_5} + \frac{7S_c\,(-4+14S_5)\,l_c}{-12+14S_5} + \frac{S_e\,(2+6S_5)\,l_e}{1+S_5}$$

$$+ l_z \equiv 2\,\alpha + l_z.$$

The coefficients of S_8 in each of the above cubics in S_8 contain the variable S_5 and otherwise only known quantities. Between the two cubics we could eliminate S_8, obtaining a relation between the coefficients which, cleared of fractions, would yield an equation of about the twentieth degree in S_5. This we could then solve directly. Or, by assuming a series of values of S_5 and determining the corresponding values of S_8 from the cubics, we can plot the curves in part for each of the above equations, and their intersections will furnish the common roots. Adopting this latter method, we find—

if $S_5 = 0$	$S^3{}_8 + 4.798S^2{}_8 + 4.764\,S_8 + 0 = 0$	has root 0
	$S^3{}_8 + 6.885S^2{}_8 + 12.385\,S_8 + 3.975 = 0$	" " −0.406
" $S_5 = -0.1$	$S^3{}_8 + 3.735S^2{}_8 + 1.912\,S_8 + 3.679 = 0$	has root −3.49
	$S^3{}_8 + 6.62\,S^2{}_8 + 10.83\,S_8 + 1.870 = 0$	" " −0.195
" $S_5 = -0.03$	$S^3{}_8 + 4.588S^2{}_8 + 4.139\,S_8 + 0.479 = 0$	has root −0 135
	$S^3{}_8 + 6.820S^2{}_8 + 12.000\,S_8 + 3.441 = 0$	" " −0.354
" $S_5 = -0.05$	$S^3{}_8 + 4.434S^2{}_8 + 3.702\,S_8 + 0.781 = 0$	has root −0.336
	$S^3{}_8 + 6.778S^2{}_8 + 11.743\,S_8 + 3.094 = 0$	" " −0.320

Plotting these, we find an intersection of the curve of the second equation with that of the first at $S_5 = -0.049$ $S_8 = -0.322$, which is

approximately the solution sought. From these values of S_5 and S_8 we get $S_7 = S_8 + 2S_5 = -0.420$.

But this gives us $\begin{matrix} S_{c_{LR}} = -\frac{1}{4} S_c (4 + 7 \times 0.420) = -\frac{1}{4} S_c \times 6.940 \\ S_{c_{rR}} = -\frac{1}{4} S_c (6 + 7 \times 0.049) = -\frac{1}{4} S_c \times 6.343 \end{matrix}$

or $S^2_{c_{LR}} > S^2_{c_{rR}}$ which is contrary to our supposition that c has its maximum stress and is designed under the loading R. All other bars have their maximum stresses as supposed. We must therefore substitute $A_c = \frac{-S_{c_{LR}}}{\sigma}$ for its previous value, rewrite our equations and solve again. This affects only one denominator of our equations (page 391), which thus changed become (the last two only being written),

$$\frac{-1}{2\sigma} \Sigma \frac{S'(S_R - S_{CR})\, l}{A} \equiv \frac{S_a(-2+7S_5)\, l_a}{4+2S_8} + \frac{7S_b(5+7S_5)\, l_b}{75+35S_5}$$

$$+ \frac{7S_c(-2+7S_5)\, l_c}{-8+14S_7} + \frac{7S_d(-2+7S_5)\, l_d}{52+14S_8} + \frac{S_e(1+3S_5)\, l_e}{1+S_5} + \frac{S_5 l_z}{S_7} = 0$$

$$\frac{-1}{2\sigma} \Sigma \frac{S'(S_{LR} - S_C)\, l}{A} \equiv \frac{S_a(3+7S_7)\, l_a}{4+2S_8} + \frac{7S_b(10+7S_7)\, l_b}{75+35S_5} + \frac{7S_c\, l_c}{2}$$

$$+ \frac{7S_d(9+7S_7)\, l_d}{52+14S_5} + \frac{S_e(2+3S_7)\, l_e}{1+S_5} + l_z = 0$$

Making the substitution $S_8 + 2S_5$ for S_7 and clearing of fractions in S_8 as before, we obtain two equations, the first a biquadratic, the second a cubic in S_8. We have

$$\begin{bmatrix} 392\,\alpha' \\ -686\, S_c\, l_c \end{bmatrix} S^4_8 + \begin{bmatrix} (1568 S_5 + 2016)\,\alpha' \\ -49(28S_5+80)\, S_c\, l_c \\ +196(-2+7S_5)\, S_a\, l_a \\ +196(-2+7S_5)\, S_d\, l_d \\ +392 S_5\, l_z \end{bmatrix} S^3_8$$

$$+ \begin{bmatrix} (1568 S^2_5 + 8512 S_5 + 1632)\,\alpha' \\ -49(160S_5+104)\, S_c\, l_c \\ +(784S_5+616)(-2+7S_5)\, S_a\, l_a \\ +7(112S_5+40)(-2+7S_5)\, S_d\, l_d \\ +(784S_5+2016)\, S_5\, l_z \end{bmatrix} S^2_8$$

$$+\begin{bmatrix}(8960\,S^2{}_5+9088\,S_5-1664)\,\alpha'\\ -10192\,S_5\,S_c\,l_c\\ +(784\,S^2{}_5+2688\,S_5-416)(-2+7\,S_5)S_a l_a\\ +7(112\,S^2{}_5+192\,S_5-32)(-2+7S_5)S_d\,l_d\\ +(4480\,S_5+1632)\,S_5\,l_z\end{bmatrix}S_8$$

$$+\begin{bmatrix}(11648\,S_5-3328)\,S_5\,\alpha'\\ +(2912\,S_5-832)\,S_5\,(-2+7\,S_5)\,S_a\,l_a\\ +7\,(224\,S_5-64)\,S_5\,(-2+7\,S_5)\,S_d\,l_d\\ +(5824\,S_5-1664)\,S_5\,l_z\end{bmatrix}=0$$

where $\alpha' \equiv \dfrac{7\,S_b\,(5+7\,S_5)\,l_b}{75+35\,S_5}+\dfrac{7\,S_c\,l_c}{2}+\dfrac{S_e\,(1+3\,S_5)\,l_e}{1+S_5}$

and

$$28\beta'\,S^3{}_8+\begin{bmatrix}160\,\beta'\\ +28\,\gamma'\\ +98\,S_a\,l_a\\ +98\,S_d\,l_d\end{bmatrix}S^2{}_8+\begin{bmatrix}208\,\beta'\\ +160\,\gamma'\\ +14(3+14\,S_5)\,S_a\,l_a\\ +14(9+14\,S_5)\,S_d\,l_d\\ +364\,S_a\,l_a\\ +196\,S_d\,l_d\end{bmatrix}S_8$$

$$+\begin{bmatrix}208\,\gamma'\\ +52\,(3+14\,S_5)\,S_a\,l_a\\ +28\,(9+14\,S_5)\,S_d\,l_d\end{bmatrix}=0$$

where $\beta' \equiv \dfrac{49\,S_b\,l_b}{75+35\,S_5}+\dfrac{3\,S_e\,l_e}{1+S_5}$ $\quad \gamma' \equiv 2\,\alpha'+l_z$.

Solving these equations just as in the preceding case we have the following:

If $S_5=-0.05$ $\quad S^4{}_8+4.394\,S^3{}_8+1.856\,S^2{}_8-1.972\,S_8-0.0309=0$ has root -0.015
$\quad\quad\quad

Applying the formulas for the bar stresses written out on page 390 we obtain the series of stresses under the various loadings shown here in figures 4*a* to 8*a* and *a* (compare figures 4 to 8, page 387). But we observe that, contrary to our suppositions, the greatest stresses in the bars *a* and *d* do not occur under *LCR* but under *CR* (and *LC*). Under this latter loading the upper chord stresses are $1\frac{1}{2}$ to $2\frac{1}{2}\%$ greater than under the full loading. The design is not, therefore, admissible, but since it is a close approximation it will be of interest to note the volume of material it takes and compare with that for the three-hinged arch. The section areas are shown in Fig. 12, below which are the bar volumes. In spite of the overstrain of the bars *a* and *d* under the loadings *CR* and *LC*, the economy is but some $4\frac{1}{2}\%$ over the three-hinged arch (see Fig. 10, page) 388.

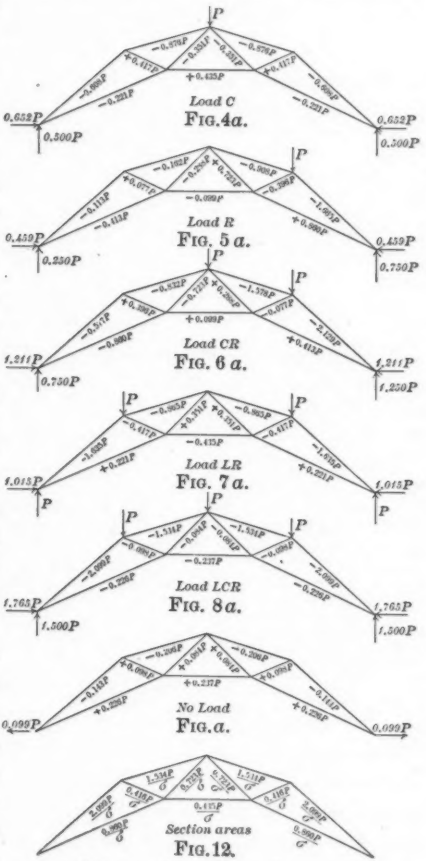


FIG. 4*a*. FIG. 5*a*. FIG. 6*a*. FIG. 7*a*. FIG. 8*a*. FIG. *a*. FIG. 12.

Bar	*a*	*b*	*c*
Volume	$16.01\frac{P}{\sigma}$	$5.11\frac{P}{\sigma}$	$2.25\frac{P}{\sigma}$

d	*e*	*z*	Total
$28.24\frac{P}{\sigma}$	$14.24\frac{P}{\sigma}$	$4.35\frac{P}{\sigma}$	$136.05\frac{P}{\sigma}$

We have now to solve on the supposition that all the bars except *c* and *z* have their greatest stresses under the loadings *CR* and *LC*, so that now

$$A_a = \frac{-S_{a_{r_{CR}}}}{\sigma} = \frac{-\frac{1}{4} S_a (12 - 7 S_5)}{\sigma}$$

$$A_d = \frac{-S_{d_{r_{CR}}}}{\sigma} = \frac{-\frac{1}{4} S_d (24 - 7 S_5)}{\sigma}.$$

We have, therefore,

$$\frac{-1}{2\sigma} \Sigma \frac{S' (S_R - S_{CR}) l}{A} \equiv \frac{7 S_a (-2 + 7 S_5) l_a}{24 - 14 S_5} + \frac{7 S_b (5 + 7 S_5) l_b}{75 + 35 S_5}$$

$$+ \frac{7 S_c (2 - 7 S_5) l_c}{8 - 14 S_7} + \frac{7 S_d (-2 + 7 S_5) l_d}{48 - 14 S_5} + \frac{S_e (1 + 3 S_5) l_e}{1 + S_5} + \frac{S_5 l_z}{S_7} = 0.$$

$$\frac{-1}{2\sigma} \Sigma \frac{S' (S_{LR} - S_C) l}{A} \equiv \frac{7 S_a (3 + 7 S_7) l_a}{24 - 14 S_5} + \frac{7 S_b (10 + 7 S_7) l_b}{75 + 35 S_5}$$

$$+ \frac{7 S_c l_c}{2} + \frac{7 S_d (9 + 7 S_7) l_d}{48 - 14 S_5} + \frac{S_e (2 + 3 S_7) l_e}{1 + S_5} + l_z = 0.$$

These contain only the variables S_5 and S_7. Writing out in terms of S_7 we obtain the equations

$$14\alpha'' S_7^2 + \begin{bmatrix} -8\alpha'' \\ -7(2 - 7 S_5) S_c l_c \\ +14 S_5 l_z \end{bmatrix} S_7 - 8 S_5 l_z = 0 \text{ and } \beta'' S_7 + \gamma'' = 0.$$

where

$$\alpha'' \equiv \frac{7 S_a (-2 + 7 S_5) l_a}{24 - 14 S_5} + \frac{7 S_b (5 + 7 S_5) l_b}{75 + 35 S_5} + \frac{7 S_d (-2 + 7 S_5) l_d}{48 - 14 S_5}$$

$$+ \frac{S_e (1 + 3 S_5) l_e}{1 + S_5}.$$

$$\beta'' \equiv \frac{49 S_a l_a}{24 - 14 S_5} + \frac{49 S_b l_b}{75 + 35 S_5} + \frac{49 S_d l_d}{48 - 14 S_5} + \frac{3 S_e l_e}{1 + S_5},$$

$$\gamma'' \equiv \frac{21 S_a l_a}{24 - 14 S_5} + \frac{70 S_b l_b}{75 + 35 S_5} + \frac{7 S_c l_c}{2} + \frac{63 S_d l_d}{48 - 14 S_5} + \frac{2 S_e l_e}{1 + S_5} + l_z.$$

Here our equations are much simpler than before, one being linear and the other quadratic in S_7. We could readily eliminate S_7, obtaining an equation of the eleventh degree in S_5, but it will be simpler to proceed as before by assuming values of S_5, computing S_7 and graphically obtaining the common roots.

If $S_5 = -0.100$ $\quad S_7^2 + 0.098495\ S_7 - 0.14068 = 0$ has root -0.4277
$\quad S_7 + 0.4345 = 0$ has root -0.4345.

If $S_5 = -0.101$ $\quad S_7^2 + 0.1087\ S_7 - 0.14349 = 0$ has root -0.4370
$\quad S_7 + 0.4348 = 0$ has root -0.4348.

Graphically interpolating we find

$S_5 = -0.1008.$ $\quad S_7 = -0.4346.$ $\quad S_8 = S_7 - 2 S_5 = -0.2330.$

We obtain through the formulas for bar stresses on page 390 the series of stresses under the various loadings shown here in Figs. 4*b* to 8*b* and *b*.

This time the greatest stresses all occur under the loads as assumed, and the solution here obtained is therefore that sought.

The section areas are as given in Fig. 13, together with the loadings under which they are designed. And below are given the volumes of the bars and the total volume (compare Fig. 10, page 388). The economy thus obtained over the three-hinged arch proves to be but 3½ per cent.

We may test the work by computing the shortening of the line z from its length when the bars a, b, c, d and e all have zero strain. We have $\frac{E}{\sigma}\delta z = \Sigma \frac{S'l}{S_{max}} S$ (if we write S_{max} always +). The values of $\frac{S'l}{S_{max}}$ being as shown in Fig. 14, we obtain, by aid of the stresses given in Figs. 4*b* to 8*b* and *b*, the values of $\frac{E}{\sigma}\delta z$ shown above.

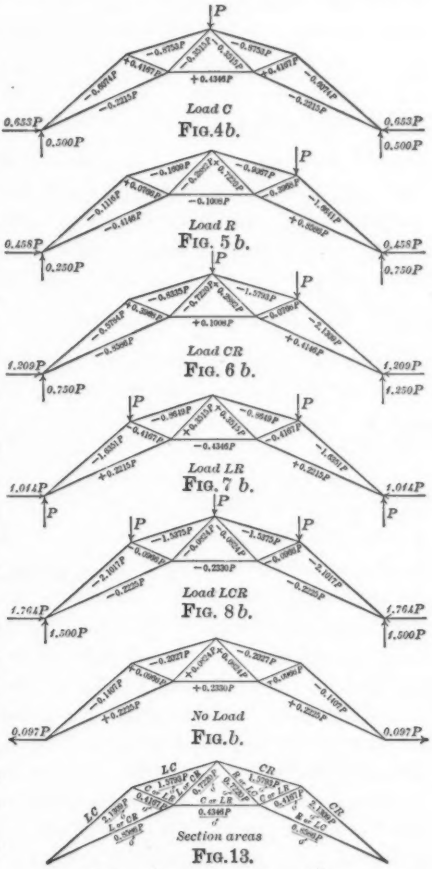


FIG. 4*b*. FIG. 5*b*. FIG. 6*b*. FIG. 7*b*. FIG. 8*b*. FIG. *b*. FIG. 13.

Bar	*a*	*b*	*c*	*d*
Volume	$16.49\frac{P}{\sigma}$	$5.11\frac{P}{\sigma}$	$2.24\frac{P}{\sigma}$	$28.67\frac{P}{\sigma}$

Bar	*e*	*z*	Total
Volume	$14.21\frac{P}{\sigma}$	$4.34\frac{P}{\sigma}$	$137.78\frac{P}{\sigma}$

Subtracting from the others the value for $\frac{1}{2}(LCR)$ when the bar z has no strain, we obtain $\frac{E}{\sigma}$ times the actual strain of z. This should be ± 10 (more accurately $\pm l_z$ which is somewhat less than 10 ft.), for loads LR and C, respectively, $\pm \frac{1008}{4346} \times 10 = \pm 2.319$ for R and CR, respectively, and $\pm \frac{2330}{4346} \times 10 = \pm 5.361$ for LCR and "no load," respectively. Even these, which are a trifle too large owing to the value of l_z used, check within $\frac{3}{10}\%$, or within the probable error from keeping but four significant figures in the calculations.

The normal or unstrained length of the bar z is its length under the loading which makes its stress zero, that is $\frac{1}{2}P$ on each panel point. This length is $(10 - 18.556 \frac{\sigma}{E}$ ft.). Taking $\sigma = 10\,000$ lbs. per square inch, and $E = 30\,000\,000$ lbs. per square inch, this correction is 0.00618 ft., the corrected length of z being 9.99382 ft. It is seen that an error of a ten-thousandth of a foot would sensibly change the strain, and the same is true of the other bars. To ensure the distribution of the stresses as here calculated, within, say, 1%, the bar lengths should all be calculated to five decimal places and single errors in the finished structure should not much exceed a unit in the last decimal place. That is, the bar lengths should be true to about one ten-thousandth of an inch; certainly closer than to one-thousandth of an inch. And we must further suppose no comparable yielding of joints or abutments.

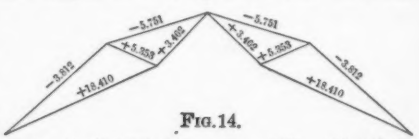


FIG. 14.

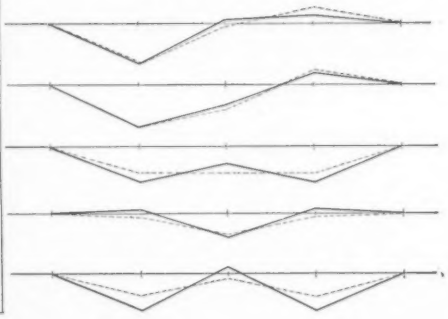
Load	C	No Load	CR	R	LCR	LR	½(LCR)
$\frac{E}{\sigma}\delta z$	+8.572	+13.198	+16.242	+20.871	+23.912	+28.540	+18.556
	−9.984	−5.358	−2.314	+2.315	+5.356	+9.984	0.000

The relative stiffness of the indeterminate design just obtained as compared with that of the determinate (three-hinged) design is a matter of much interest, since it is frequently claimed that the two-hinged arch is stiffer than the three-hinged arch. If by S' we indicate the stresses in the three-hinged arch due to a vertical load of unity on any joint and by S the actual stresses in the bars under any given loading, in either design, then $\Sigma \frac{S'Sl}{EA}$ will give the sinking of the joint in question under the given loading. The values of S' for load

unity on the middle joint or on the right upper chord joint we readily obtain from Figs. 4 and 5, the values of S we obtain from Figs. 4 to 8 and Figs. 4b to 8b and b, and the values of A from Fig. 10 and Fig. 13. We note that for the indeterminate design "no load" also gives deflections, $-3.72\ \frac{\sigma}{E}$ at the middle joint and $+6.19\ \frac{\sigma}{E}$ at the side upper chord joints. The deflections, corrected as above for the indeterminate design and without the factor $\frac{\sigma}{E}$, are as follows:

Load	Joint	Deflection; Three-hinged arch.	Deflection; Two-hinged arch.
L	*L*	+49.96	+44.64
	C	− 2.63	+ 2.52
	R	− 8.84	−18.40
LC	*L*	+47.33	+47.18
	C	+22.29	+26.12
	R	−11.47	−15.86
LCR	*L*	+38.50	+28.79
	C	+19.66	+28.64
	R	+38.50	+28.79
C	*L*	− 2.63	+ 2.54
	C	+24.92	+23.59
	R	− 2.63	+ 2.54
LR	*L*	+41.13	+26.24
	C	− 5.25	+ 5.05
	R	+41.13	+26.24

Plotted Deflections:
Full line for Three-hinged arch.
Dotted " " Two- " "



We note that the deflections differ in character in the two cases. The greatest deflections do not occur under the same loadings. The greatest side joint deflection of the three-hinged arch is some 6% greater than that of the two-hinged arch, but the greatest center joint deflection of the three-hinged arch is some 15% less than that of the two-hinged arch. The center deflections are little more than half the quarter deflections in both cases. Practically one arch is about as stiff as the other. Certainly the existence of the middle bar in the two-hinged arch does not appear to have given it a pronounced advantage.

Is the indeterminate solution which we have found one of greatest economy of material? It at least is a solution at a point of discontinuity in the variation of the material used. That is, under our condition of equal maximum stress, if the compression in z under R were slightly less, the tension in z under $C\,R$ would be slightly greater, A_b and A_e would be determined under the loading R, and A_c and A_z would be determined under the loading C. But if the compression

in z under R were slightly greater, the tension in z under $C\,R$ would be slightly less, A_b and A_e would be determined under the loading $C\,R$ and A_c and A_z would be determined under the loading $L\,R$. For, consider the three equations between R and $C\,R$, $L\,R$ and C, and R and $L\,R$ in the former case. Put $S_{z_R} = S_5 + \Delta_5$, $S_{z_{CR}} = -S_5 + \Delta_6$, $S_{z_{LR}} = S_7 + \Delta_7$ and $S_{z_C} = -S_7 + \Delta_4$, where Δ_5 is $+$. Then

$$-\frac{1}{\sigma}\Sigma\frac{S'(S_R - S_{CR})\,l}{A}$$

$$\equiv \frac{7\,S_a(-4+14\,S_5+7\Delta_5-7\Delta_6)\,l_a}{24-14\,S_5+14\Delta_6}$$

$$+\frac{7\,S_b(10+14\,S_5+7\Delta_5-7\Delta_6)\,l_b}{75+35\,S_5+35\Delta_5}$$

$$+\frac{7\,S_c(4-14\,S_5-7\Delta_5+7\Delta_6)\,l_c}{8-14\,S_7+14\Delta_4}$$

$$+\frac{7\,S_d(-4+14\,S_5+7\Delta_5-7\Delta_6)\,l_d}{48-14\,S_5+14\Delta_6}$$

$$+\frac{S_e(2+6\,S_5+3\Delta_5-3\Delta_6)\,l_e}{1+S_5+\Delta_5}$$

$$+\frac{(2\,S_5+\Delta_5-\Delta_6)\,l_z}{S_7-\Delta_4}$$

$$=0$$

$$-\frac{1}{\sigma}\Sigma\frac{S'(S_{LR} - S_C)\,l}{A}$$

$$\equiv \frac{7\,S_a(6+14\,S_7+7\Delta_7-7\Delta_4)\,l_a}{24-14\,S_5+14\Delta_6}$$

$$+\frac{7\,S_b(20+14\,S_7+7\Delta_7-7\Delta_4)\,l_b}{75+35\,S_5+35\Delta_5}$$

$$+\frac{7\,S_c(8-14\,S_7-7\Delta_7+7\Delta_4)\,l_c}{8-14\,S_7+14\Delta_4}$$

$$+\frac{7\,S_d(18+14\,S_7+7\Delta_7-7\Delta_4)\,l_d}{48-14\,S_5+14\Delta_6}$$

$$+\frac{S_e(4+6\,S_7+3\,\Delta_7-3\,\Delta_4)\,l_e}{1+S_5+\Delta_5}$$

$$+\frac{(2S_7+\Delta_7-\Delta_4)\,l_z}{S_7-\Delta_4}$$

$$=0$$

$$-\frac{1}{\sigma}\Sigma\frac{S'(S_{LR} - S_R)\,l}{A}$$

$$\equiv \frac{7\,S_a(5+7\,S_7-7\,S_5+7\Delta_7-7\,\Delta_5)\,l_a}{24-14\,S_5+14\Delta_6}$$

$$+\frac{7\,S_b(5+7\,S_7-7\,S_5+7\,\Delta_7-7\,\Delta_5)\,l_b}{75+35\,S_5+35\,\Delta_5}$$

$$+\frac{7\,S_c(2-7\,S_7+7\,S_5-7\Delta_7+7\,\Delta_5)\,l_c}{8-14\,S_7+14\,\Delta_4}$$

$$+\frac{7\,S_d(11+7\,S_7-7\,S_5+7\,\Delta_7-7\,\Delta_5)l_d}{48-14\,S_5+14\,\Delta_6}$$

$$+\frac{S_e(1+3\,S_7-3\,S_5+3\,\Delta_7-3\,\Delta_5)\,l_e}{1+S_5+\Delta_5}$$

$$+\frac{(S_7-S_5+\Delta_7-\Delta_5)\,l_z}{S_7-\Delta_4}$$

$$=0$$

Remembering that the Δ are all very small, and noting the two equations on page 396 and also the equation formed by their difference, the equations just written may be reduced to:

$$-\frac{1}{\sigma}\Sigma\frac{S'(S_R-S_{CR})\,l}{A} \qquad -\frac{1}{\sigma}\Sigma\frac{S'(S_{LR}-S_C)\,l}{A}$$

$$=\frac{7S_a(7\Delta_5-7\Delta_6+\frac{4-14S_5}{24-14S_5}\times 14\,\Delta_6)\,l_a}{24-14S_5} \qquad =\frac{7S_a(7\Delta_7-7\Delta_4-\frac{6+14S_7}{24-14S_5}\times 14\Delta_6)\,l_a}{24-14S_5}$$

$$+\frac{7S_b(7\Delta_5-7\Delta_6-\frac{10+14S_5}{75+35S_5}\times 35\Delta_5)\,l_b}{75+35S_5} \qquad +\frac{7S_b(7\Delta_7-7\Delta_4-\frac{20+14S_7}{75+35S_5}\times 35\Delta_5)\,l_b}{75+35S_5}$$

$$+\frac{7S_c(-7\Delta_5+7\Delta_6-\frac{4-14S_5}{8-14S_7}\times 14\Delta_4)\,l_c}{8-14S_7} \qquad +\frac{7S_c(-7\Delta_7+7\Delta_4-14\Delta_4)\,l_c}{8-14S_7}$$

$$+\frac{7S_d(7\Delta_5-7\Delta_6+\frac{4-14S_5}{48-14S_5}\times 14\Delta_6)\,l_d}{48-14S_5} \qquad +\frac{7S_d(7\Delta_7-7\Delta_4-\frac{18+14S_7}{48-14S_5}\times 14\Delta_6)\,l_d}{48-14S_5}$$

$$+\frac{S_e(3\Delta_5-3\Delta_6-\frac{2+6S_5}{1+S_5}\times\Delta_5)\,l_e}{1+S_5} \qquad +\frac{S_e(3\Delta_7-3\Delta_4-\frac{4+6S_7}{1+S_5}\times\Delta_5)\,l_e}{1+S_5}$$

$$+\frac{(\Delta_5-\Delta_6+\frac{2S_5}{S_7}\times\Delta_4)\,l_z}{S_7} \qquad +\frac{(\Delta_7-\Delta_4+2\Delta_4)\,l_z}{S_7}$$

$$=0 \qquad =0$$

$$-\frac{1}{\sigma}\Sigma\frac{S'\,(S_{LR}-S_R)\,l}{A}$$

$$=\frac{7S_a(7\Delta_7-7\Delta_5-\frac{5+7S_7-7S_5}{24-14S_5}\times 14\Delta_6)\,l_a}{24-14S_5}$$

$$+\frac{7S_b\,(7\Delta_7-7\Delta_5-\frac{5+7S_7-7S_5}{75+35S_5}\times 35\Delta_5)\,l_b}{75+35S_5}$$

$$+\frac{7S_c(-7\Delta_7+7\Delta_5-\frac{2-7S_7+7S_5}{8-14S_7}\times 14\Delta_4)\,l_c}{8-14S_7}$$

$$+\frac{7S_d(7\Delta_7-7\Delta_5-\frac{11+7S_7-7S_5}{48-14S_5}\times 14\Delta_6)\,l_d}{48-14S_5}$$

$$+\frac{S_e(3\Delta_7-3\Delta_5-\frac{1+3S_7-3S_5}{1+S_5}\times\Delta_5)\,l_e}{1+S_5}$$

$$+\frac{(\Delta_7-\Delta_5+\frac{S_7-S_5}{S_7}\times\Delta_4)\,l_z}{S_7}$$

$$=0.$$

These are three linear equations between the four quantities Δ_4, Δ_5, Δ_6 and Δ_7.

Putting in the numerical values of S_a, S_b, S_c, S_d, S_e, l_a, l_b, l_c, l_d, l_e, l_z, and of S_5 and S_7, these equations may be reduced to the numerical equations:

$$\left.\begin{aligned} &\Delta_4 + 4.26\Delta_5 - 5.29\Delta_6 = 0 \\ &\Delta_4 + 0.82\Delta_5 + 0.09\Delta_6 - 3.21\Delta_7 = 0 \\ &\Delta_4 - 3.92\Delta_5 - 0.18\Delta_6 + 3.90\Delta_7 = 0 \end{aligned}\right\} \text{ which give } \left\{\begin{aligned} \Delta_5 &= +1.4\Delta_7 \\ \Delta_6 &= +1.5\Delta_7 \\ \Delta_4 &= +1.9\Delta_7 \end{aligned}\right.$$

Thus the Δ are all + as stated. But this change then increases all of the designing stresses $S_{a_{r_{CR}}}$, $S_{b_{r_R}}$, S_{c_C}, $S_{d_{r_{CR}}}$, $S_{e_{r_R}}$, and S_{z_C}, and therefore increases the material used.

Now, consider the similar equations between the loadings R and CR, LR and C, and R and LR in the second case, when Δ_5 is —, and a, b, d and e are designed under CR, and c and z under LR. We may at once write them down in the second form, the change being simply replacing Δ_5 by $-\Delta_6$ and Δ_4 by $-\Delta_7$ where, in the brackets, these have a fraction for a coefficient. We have:

$$-\frac{1}{\sigma}\Sigma\frac{S'(S_R - S_{CR})l}{A}$$

$$= \frac{7S_a(7\Delta_5 - 7\Delta_6 + \frac{4-14S_5}{24-14S_5}\times 14\Delta_6)\,l_a}{24-14S_5}$$

$$+ \frac{7S_b(7\Delta_5 - 7\Delta_6 + \frac{10+14S_5}{75+35S_5}\times 35\Delta_6)l_b}{75+35S_5}$$

$$+ \frac{7S_c(-7\Delta_5 + 7\Delta_6 + \frac{4-14S_5}{8-14S_7}\times 14\Delta_7)l_c}{8-14S_7}$$

$$+ \frac{7S_d(7\Delta_5 - 7\Delta_6 + \frac{4-14S_5}{48-14S_5}\times 14\Delta_6)l_a}{48-14S_5}$$

$$+ \frac{S_e(3\Delta_5 - 3\Delta_6 + \frac{2+6S_5}{1+S_5} + \Delta_6)\,l_e}{1+S_5}$$

$$+ \frac{(\Delta_5 - \Delta_6 - \frac{2S_5}{S_7} + \Delta_7)\,l_z}{S_7}$$

$$= 0$$

$$-\frac{1}{\sigma}\Sigma\frac{S'(S_{LR} - S_C)\,l}{A}$$

$$= \frac{7S_a(7\Delta_7 - 7\Delta_4 - \frac{6+14S_7}{24-14S_5}\times 14\Delta_6)\,l_a}{24-14S_5}$$

$$+ \frac{7S_b(7\Delta_7 - 7\Delta_4 + \frac{20+14S_7}{75+35S_5}\times 35\Delta_6)\,l_b}{75+35S_5}$$

$$+ \frac{7S_c(-7\Delta_7 + 7\Delta_4 + 14\Delta_7)\,l_c}{8-14S_7}$$

$$+ \frac{7S_d(7\Delta_7 - 7\Delta_4 - \frac{18+14S_7}{48-14S_5}\times 14\Delta_6)\,l_a}{48-14S_5}$$

$$+ \frac{S_e(3\Delta_7 - 3\Delta_4 + \frac{4+6S_7}{1+S_5}\times\Delta_6)\,l_e}{1+S_5}$$

$$+ \frac{(\Delta_7 - \Delta_4 - 2\Delta_7)\,l_z}{S_7}$$

$$= 0$$

$$-\frac{1}{\sigma}\Sigma\frac{S'(S_{LR}-S_R)\,l}{A}$$

$$=\frac{7S_a\,(7\Delta_7-7\Delta_5-\dfrac{5+7S_7-7S_5}{24-14S_5}\times 14\Delta_6)\,l_a}{24-14S_5}$$

$$+\frac{7S_b\,(7\Delta_7-7\Delta_5+\dfrac{5+7S_7-7S_5}{75+35S_c}\times 35\Delta_6)\,l_b}{75+35S_5}$$

$$+\frac{7S_c\,(-7\Delta_7+7\Delta_5+\dfrac{2-7S_7+7S_5}{8-14S_7}\times 14\Delta_7)\,l_c}{8-14S_7}$$

$$+\frac{7S_d\,(7\Delta_7-7\Delta_5-\dfrac{11+7S_7-7S_5}{48-14S_5}\times 14\Delta_6)\,l_d}{48-14S_5}$$

$$+\frac{S_e(3\Delta_7-3\Delta_5+\dfrac{1+3S_7-3S_5}{1+S_5}\times\Delta_6)\,l_e}{1+S_5}$$

$$+\frac{(\Delta_7-\Delta_5-\dfrac{S_7-S_5}{S_7}\times\Delta_7)\,l_z}{S_7}$$

$$=0$$

These give three linear equations between the four quantities Δ_4, Δ_5, Δ_6 and Δ_7, which may be reduced to the numerical equations:

$$\left.\begin{aligned}\Delta_5-0.687\Delta_6-0.177\Delta_7&=0\\ \Delta_4-0.230\Delta_6-0.313\Delta_7&=0\\ \Delta_5+0.041\Delta_6-0.744\Delta_7&=0\end{aligned}\right\}\text{ which give....}$$

$$\text{.....}\left\{\begin{aligned}\Delta_6&=+0.78\Delta_7\\ \Delta_5&=+0.71\Delta_7\\ \Delta_4&=+0.49\Delta_7\end{aligned}\right.\qquad \Delta_7=\frac{+1}{0.78}\qquad \Delta_6=+1.28\Delta_6$$

Thus the Δ are all of the same sign, or negative, and a, b, d and e have their greatest stresses under CR, and c and z theirs under LR, as stated. The changes in volume due to a change Δ_6 (increased compression in z under CR) are:

Volume of b increases $S'_b l_b \Delta_6 \frac{1}{\sigma} = 2.50 \Delta_6 \frac{1}{\sigma}$

" " e " $S'_e l_e \Delta_6 \frac{1}{\sigma} = 15.80 \Delta_6 \frac{1}{\sigma}$

" " c " $S'_c l_c \Delta_7 \frac{1}{\sigma} = 2.85 \Delta_6 \frac{1}{\sigma}$

$21.15 \Delta_6 \frac{1}{\sigma}$

2

" " $2(b + c + e)$ increases $= 42.30 \Delta_6 \frac{1}{\sigma}$

" " z " $l_z \Delta_7 \frac{1}{\sigma} = 12.80 \Delta_6 \frac{1}{\sigma}$

" " $2(b + c + e) + z$ increases $55.10 \Delta_6 \frac{1}{\sigma}$

Volume of a diminishes $S'_a l_a \Delta_6 \frac{1}{\sigma} = 9.08 \Delta_6 \frac{1}{\sigma}$

" " d " $S'_d l_d \Delta_6 \frac{1}{\sigma} = 8.12 \Delta_6 \frac{1}{\sigma}$

$17.20 \Delta_6 \frac{1}{\sigma}$

2

" " $2(a + d)$ diminishes $= 34.40 \Delta_6 \frac{1}{\sigma}$

Thus this variation also increases the material used, the saving of material on the bars a and d being considerably more than offset by the increased material in the bars b, c, e and z. Thus the indeterminate solution we have found is at a point of discontinuity which is a point of greatest economy. Now it is readily shown that a larger primary stress in z is always less economic, and simple investigation shows that the smaller the primary tension in z and therefore the greater its greatest compression, the less saved on the bars a and d compared with that expended on the bars b, c, e and z. This solution is therefore the most economic symmetric solution; and as investigation shows that unsymmetric solutions are less economic than the symmetric, we may conclude that the design obtained is the most economic solution of the given two-hinged arch for the given loadings, on the basis of a uniform greatest mean stress in designing.

Now, the economic superiority of this most economic indeterminate design over the simple determinate design (the three-hinged arch), is but some 3½% (see page 397). Such a small saving, if all other things were equal, would hardly be worth the labor of seeking; but we recognize that, aside from the difficulty of determining the design, other things are not at all equal, for, to actually realize the stress distribution calculated, so accurately as to really preserve the 3½% advantage, extremely accurate and therefore expensive work would be necessary, and also very rigid and therefore expensive abutments. Then, once constructed, no set may occur under use, or conditions would not be maintained as supposed; and last, but not least, changes in temperature are not allowable since they would greatly change the stresses. Thus, with $E = 30\,000\,000$ lbs. per square inch and $\sigma = 10\,000$ lbs. per square inch, a change of 0.064 ft. in the span would cause a stress of P in the bar z, and the corresponding stresses in the remaining bars (see Fig. 3 or Fig. 9). With a coefficient of expansion of $\frac{6\frac{1}{2}}{1\,000\,000}$ per degree Fahrenheit, 100° Fahr. would correspond to a change in span of $\frac{6\frac{1}{2} \times 100 \times 40}{1\,000\,000} = 0.4 \times 0.065$ ft., or $\frac{4}{10}$ P stress in z, nearly. This result is shown in Fig. 15. And while 100° Fahr. change from the mean (that is 200° Fahr. range of temperature), may be excessive, even if we take only half that or 50° Fahr. change (100° Fahr. range), we get temperature stresses of half those shown in Fig. 15 which, by comparison with those in Fig. 13, we see are some 45% of the greatest stress in z, 22% of that in e, 20% of that in c, 11% of that in a, 10% of that in b, and 6% of that in d. That these stresses, absent from the three-hinged arch, turn the balance completely in its favor, many times offsetting the 3½% economy in the two-hinged arch found by neglecting temperature changes, is self-evident to any impartial student of the subject.

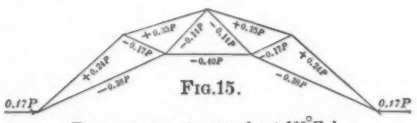


FIG. 15.

Temperature stresses for +100° Fahr.

Finally, we may note that even the 3½% theoretic advantage of the two-hinged arch can be exceeded without the use of indeterminate construction through the use of what has elsewhere been termed "multiple construction." Thus we have the three-hinged arch design

of Fig. 16, capable of supporting all the loadings except $L\,C\,R$ with stress σ or less (see Figs. 4, 5, 6 and 7). For the load $L\,C\,R$ we introduce a bar z of area $\frac{2}{7}\frac{P}{\sigma}$, give the junction of the bars a_l and a_r freedom to move vertically with respect to the junction of the bars b_l and b_r, and load the junction of $a_l a_r$ with $\frac{5}{7}P$, and the junction of $b_l b_r$ with the remainder or $\frac{2}{7}P$. The resulting stresses in this framework, which is statically determined, are shown in Fig. 17. We see that these stresses are equal to or less than the designing stresses shown in Fig. 16. The construction permitting of this might be any such as would ordinarily leave z unstressed, but, as by the insertion of a wedge, cause it to take compression when desired, while the joint of $a_l a_r$ could at the same time be freed from the joint of $b_l b_r$, say by removing another wedge. But it suffices to note that the construction is possible; with the mechanical details we are not here concerned.

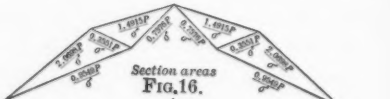


FIG. 16.

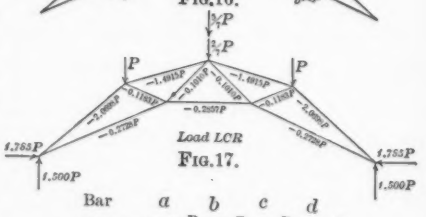


FIG. 17.

Bar	a	b	c	d
Volume	$15.57\frac{P}{\sigma}$	$5.36\frac{P}{\sigma}$	$1.91\frac{P}{\sigma}$	$27.85\frac{P}{\sigma}$
Bar	e	z	Total	
Volume	$15.81\frac{P}{\sigma}$	$2.86\frac{P}{\sigma}$	$135.84\frac{P}{\sigma}$	

The volume of material here, as shown under Fig. 17, is some $1\frac{1}{2}\%$ less than that required by the most economic indeterminate design. Very possibly other multiple determinate designs might be found with still somewhat more economy of material. But with these we need not concern ourselves, the essential point being simply that, as has been stated previously, there will be some statically determined method of solution more economic than any indeterminate solution, even on the theoretic conditions most favorable to the latter. It is not intended to intimate, however, that these multiple determinate solutions are in general desirable substitutes for the simpler but less economic single determinate solutions. The multiple solutions present more or less considerable mechanical difficulties in their adaptation to practical use, and while these difficulties are not

insuperable, in fact, perhaps, are less considerable than the exact construction of an indeterminate framework, they would probably prevent their use where the saving was not considerable. In a case like the present, where the saving would be but some 5%, it is hardly likely that they would be used.

This simple example of the indeterminate arch, although valueless as a design because of the neglect of post formulas, as a comparative study, is affected little by such considerations, and illustrates very well certain points in economic design. While, theoretically, the indeterminate framework may give a more economic design for several loadings, than does any single determinate framework of included figure, the amount of such theoretic advantage is considerable only in most extraordinary cases; and there it would probably pay to use a multiple determinate design which would be still more economic. In all cases where the figure of the simple determinate design closely approximates to the figure of the indeterminate design the theoretic advantage possible through using the latter will certainly be insignificant. Actually, owing to temperature stresses, etc., that advantage will not exist at all. Again, as the structure becomes larger, and the variations of stress due to variable loadings affect smaller and smaller parts of the structure to a sensible degree, the possible economy of indeterminate construction proportionally diminishes. It may safely be assumed that in actual structures (as arches and suspension bridges), a theoretic advantage of even 5% cannot exist in the most economic indeterminate design over the most economic determinate design of included figure, even when temperature stresses are neglected. It is doubtful if ordinarily 1% economy could be found; and temperature stresses alone would offset such advantages several times over in all cases. This illustration, then, definitely confirms the view that the use of indeterminate frameworks is certainly not advantageous from an economic standpoint, nor apparently from any other standpoint.

DISCUSSION.

Mr. Goldmark. HENRY GOLDMARK, M. Am. Soc. C. E. — In this interesting paper, the author applies the well-known principle of Virtual Velocities to a study of the stresses and deflections in trusses with redundant members and other structures not statically determinate.

As the result of his investigation, he arrives at the conclusion that such constructions show no advantage, either in economy of material or in rigidity, over the simpler determinate forms, while even the best methods for computing their stresses are extremely laborious and of doubtful accuracy.

He therefore recommends that all indeterminate forms, such as trusses with a multiple web system, continuous girders, arches with less than three hinges, suspension bridges with continuous stiffening girders, etc., should be abandoned entirely in practice.

As to the method of computation which he has used (that of Virtual Velocities or Least Work), he considers the amount of numerical labor involved so great as to make the method practically valueless.

While the speaker is by no means an advocate of complex truss forms where there is no good reason for using them, he is fully satisfied that there are many cases in practice where these tabooed forms give by far the best results.

As a matter of fact, the simplicity of the strain sheet is, in many cases, by no means a proper criterion as to the excellence of the design. Thus, in railroad bridges of moderate length, the tendency of late years, as the result of actual experience, is away from the "ideal" single-intersection bridge with alleged frictionless pins. Such spans are now almost universally built with riveted connections, and, in many cases, with multiple systems of webbing. Even in larger bridges, the floor system is almost invariably riveted between the posts, though the transference of loads to the main trusses is thus made less direct. In all these cases intricate secondary stresses are introduced, but the advantages obtained are believed to counterbalance this theoretical drawback.

In the case of the larger arches and suspension bridges the introduction of hinges at the center is properly considered to break the continuity of the floor system and wind bracing, to add to the complication of the details, and in all probability to decrease the rigidity of the whole bridge under moving loads.

It is true that the absence of hinges makes the computation of stresses more difficult and somewhat less certain, but it is believed that with the homogeneous material now used, and the excellent workmanship of our shops, perfectly safe structures can be built without any

excessive cost. Besides this, there is, in the speaker's mind, little doubt that even the best designed hinges or pins can hardly be counted upon to act as theory requires. Mr. Goldmark.

But besides such bridge forms as those above mentioned, in which the designer is at liberty to choose between so-called determinate and indeterminate forms, there are many engineering structures which are, of necessity, of a complex nature. Such are the frame works for the enclosing of large shops and auditoriums, in which the roof trusses and the steel posts which support them are connected by knee-braces or other connections, and many forms of cranes and derricks; furthermore, metallic gates for canal locks, as well as ship caissons for dry docks and harbor works. In all these constructions some method of computing indeterminate stresses is desirable.

In such cases, as well as for computing bridge stresses, the speaker has found the Method of Virtual Displacements (Method of Least Work) of great value, while he has not found it excessively laborious, particularly if it is used as a means of devising approximate methods of sufficient exactness in each case rather than for computing individual designs.

A brief reference to the bibliography of this method of computation may be of some interest.

The first application of the principle of Virtual Velocities to the determination of deflections and stresses in frameworks is commonly ascribed to Clerk Maxwell, whose paper embodying this method appeared in the *Philosophical Magazine* for 1864. The same theorem, in a slightly different form, was, however, used independently, to solve problems of this class, by M. Menébrea,* in 1858, and Professor Lamé,† in 1866.

The further systematic development of the method is, however, due mainly to German writers. The earlier articles of Professors Mohr‡ and Winkler§ gave an application of the method to a variety of structures, such as continuous girders, trusses with multiple web systems, arches with less than three hinges, etc. The most complete exposition of the more recent German methods, however, is given in the works of Professor H. Müller-Breslau, of Berlin. His special treatise‖ on the subject the speaker has found of great assistance in practice, and he would consider an English translation of much value for American engineers.

Reference should also be made to a monograph¶ in French, by the late M. Castigliano, a young Italian engineer of great promise, who

* *Comptes Rendus*, 1858, Vol. xlvi, p. 1056.

† "Leçons sur la Théorie Mathematique d'Elasticité des Corps Solides."

‡ *Zeitschrift des Architekten und Ingenieur Vereins zu Hannover*, 1874 and 1875.

§ *Zeitschrift des Architekten und Ingenieur Vereins zu Hannover*, 1879.

‖ "Die neueren Methoden der Festigkeitslehre." Berlin.

¶ "Théorie de l'Équilibre des Systèmes Élastiques." Turin, 1879.

Mr. Goldmark. gives a very elegant analysis of the "Method of Least Work," with many applications.

In the English language, the valuable paper by George F. Swain, M. Am. Soc. C. E., published in 1883,* is the only extended article on this subject known to the speaker, though brief expositions are given in some textbooks on applied mechanics.† Mr. Cilley has therefore conferred an obligation, on English-speaking engineers, by his interesting study on the subject.

Mr. Lindenthal. GUSTAV LINDENTHAL, M. Am. Soc. C. E. (by letter).—This paper is a contribution to the old controversy as to whether or not statically determinate structures are superior to statically indeterminate ones, and is a scholarly attempt on the affirmative side of the question.

Mr. Cilley, like others before him, advances as principal reasons in favor of determinate designs, their alleged economy, the greater certainty of their calculation, and the greater accuracy, therefore, in the dimensioning of the cross-sections.

Granting, that the last two arguments have much to commend them, it is nevertheless true, as regards economy, that diminutive differences in weight of metal, as shown by strain sheets of one and another type, will rarely be the sole deciding element in the choice of designs. Other considerations than mere economy of metal influence greatly the cost and merit of a structure. But in any case, strain sheets to be comparable should be complete, and include all strains which affect the sections. This does not seem to the writer to be the case with the strain diagrams in the paper. Before indicating in what the incompleteness is believed to consist, the preliminary question is justified:

"On what theoretically equivalent basis (aside from same length of span and moving load), should a comparison of the economy of determinate with indeterminate arch types be made? Should it be on the basis of the same figure of inclusion, as the author has done, or on the basis of congruous equilibrium polygons under some assumed load, as would undoubtedly be the case for ribbed arches, or for suspension bridges, which are nothing but inverted arches?"

The short span of primitive figure, chosen by the author with only 10, respectively 11, members in the frame, serves to illustrate the great labor of a detailed analysis for each member of the indeterminate arch; but in all other respects it is misleading. If the diagrams of the two types were each enlarged from 4 to, say, 40 panels, then the determinate figure of the author would become the form known as the Eads arch, for which a certain depth at the quarter is economic, and the indeterminate figure would become a sickle arch, for which another depth at the center is economic. The enlarged diagrams, however, would no longer have the same figure of inclusion. Only one chord,

* *Journal*, Franklin Institute, February, March and April, 1883.

† Cotterill, "Applied Mechanics," p. 551.

namely the upper one, would coincide in both, but that is not enough to furnish equivalent conditions of economy. Mr. Lindenthal.

Suppose a comparison were made between single (determinate) and continuous (indeterminate) trusses. Obviously, we would start out with the same ratio of height to span (at center of span) for each kind, and compare the strain sheets and material required for them on that basis, to come to a correct conclusion of their economy. Similarly, a comparison of arches will have to proceed. We cannot compare fairly the economic merits of arches of different rise, or of an arch having a ratio of rise to span of three-tenths with another arch having the same ratio as to one chord only and another much smaller ratio as to the lower chord, as Mr. Cilley does. He compares two primitive figures, one statically determinate, and the other made indeterminate by inserting a redundant member. Considered as generic forms, the figures are dissimilar and not comparable. His conclusions as to the economy, or the want of it, in indeterminate forms, arrived at in this manner, are wholly irrelevant and inapplicable to long spans, the only ones for which arch types are economically used.

This will appear more clearly when we consider that the closed frame of the author, with a depth of one-eighth of the span at the center would in practice occur as a sickle truss which is determinate, but hardly as a sickle arch which is indeterminate, and with that depth of rib, would be wasteful. No arch bridge has been built yet with a depth of rib of more than one-sixteenth. More frequently it is less than one-thirty-second of the span.

As the temperature strains for the same arch form diminish directly with the ratio of rib to span, they are, in existing bridges, very much smaller than those given by Mr. Cilley on page 405. His maximum is $\pm$ 45% of the full load for $\pm$ 100° Fahr. In the largest sickle arches thus far built, namely, that of the Garabit Viaduct in France and of the Gruenenthal Bridge over the Baltic Canal in Germany, the maxima are under $\pm$ 18%, when corrected to the same extremes of temperature.

But, as stated before, a comparison between forms of the general characteristics of sickle arches and Eads arches, simply because they happen to coincide as to one chord, can lead to no rational conclusions.

Comparisons of carefully worked out designs of different arch types, made by him and others, have convinced the writer that, other things being equal, the indeterminate types are invariably more economical. This is true even if the assumption that there are no temperature strains in three-hinged types were correct. But this assumption is not correct as shown by the writer long ago.* That time-honored theory, found in all text books, is based on error. Three-hinged arches are not

* *Engineering News*, 1888, p. 174; and "Appendix to Report of Board of Engineer Officers as to Maximum Span Practicable for Suspension Bridges," 1894, p. 68, and elsewhere.

Mr. Lindenthal. only not free from temperature strains, but for certain conditions such strains are just as large as in arches of the two-hinged type. The difference is merely in their distribution. The supposed advantage in that respect of the three-hinged over the two-hinged arch is illusory.

Professors Merriman and Jacoby's "Higher Structures" is the only work, so far, referring to the writer's theory on this subject. But although mentioned there merely in connection with three-hinged stiffening trusses in suspension bridges, it applies, nevertheless, to all forms of arches with three hinges, whether of the ribbed, spandrel-braced, or of the erect or suspended form. Each is, necessarily, differently affected; but in all cases the strains are due to the changes in the curvature of the equilibrium curve from dead load for different temperatures.

If the author will trace the strains, due to the falling and rising of the center hinge from changes of temperature, he will find them by no means of negligible smallness, even for the large ratio of rise to span in his inconclusive figure.

The temperature strains are particularly large in flat arches, and, to repeat it, the insertion of a center hinge does not eliminate them. It only serves to remove the maximum bending moment from the center to each quarter. So that where the two-hinged arch has one maximum bending moment, the three-hinged has two of the same intensity in most cases.

The new Alexander III Bridge in Paris, for instance, having a rise of only one-seventeenth of the span, is thus far the flattest metal arch attempted. It is of the three-hinged type, here properly chosen because a small depth of rib was necessitated at the center by the prescribed clearance above the river. The writer has not seen a strain sheet of that bridge, but believes that in accordance with the common theory, the arches have been assumed to be free from temperature strains. As a matter of fact, the bending moments at the quarters, from temperature changes, are fully as great as at the center of a two-hinged arch, having the same depth throughout as at the quarters (where the rib is one seventy-second of the span).

If the Washington Bridge in New York (510-ft. span), or the Niagara Falls road bridge (840-ft. span), which are both of the ribbed-arch type, had been built with three hinges instead of only two, the extra metal required to meet the temperature strains would have been the same as for two-hinged arches, with this difference, that the additions to the chord sections would have been largest at the quarters instead of at the center, and the additions to the web members largest at the ends of each half arch, instead of at the ends of the whole arch. In spandrel-braced arches, the center hinge may be of value, as here it really reduces the intensity of temperature strains, but does not eliminate them, by any means.

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The law applies also to stiffened suspension bridges. The well-known Point Bridge at Pittsburg (800-ft. span) has three hinges; the bracing is above the chains. The bridge has been believed to be free from temperature strains, which is not the case. The maxima from temperature are those corresponding to the maxima at the center of a suspended two-hinged arch, having a rise of one-eighth of the span, and a depth of rib of one thirty-second of the span. But in this case the maxima in the chords, which in the two-hinged arch occur only at the middle of the span, reach, with little variation, from hinge to hinge, while the web is only slightly affected. On the whole, this type, and the so-called Fidler type, are the worst forms for temperature strains, and yet both are determinate forms. The suspended end spans of the Tower Bridge in London, which are of the Fidler type, are certainly exposed to bending strains from temperature changes, although the designers believed them to be eliminated by the insertion of middle hinges.

That the neglect to provide for temperature strains in three-hinged arches has not yet endangered such bridges proves anew how useful the so-called factor of safety is, and how true its designation as a factor of ignorance.

A second correction in the strain sheets, for a proper comparison of the two types treated by Mr. Cilley, is required for the large bending moments resulting from the immobility of the hinges, incorrectly ignored by him. In none of the existing arch bridges has a turning motion at the hinge ever been observed. The friction is too great. But the motion is not even desirable. There could be no motion without wear, which after a while might affect the strains more than the immobility of the hinges.

The bending moments from that cause cannot be regarded as mere secondary strains of local effect, because they make themselves felt throughout the entire arch frame, and in flat arches or shallow ribs require considerable additions to the sections. It is true that the moments can be much reduced by using for the hinges small and long pins rather than short and large ones, for affording the necessary bearing area. But in any event there are, under equivalent conditions, twice as many bending moments from that cause in three-hinged as in two-hinged arches, while they are absent entirely in hingeless arches, in which the temperature strains are largest.

The present theory of metal arches requires, therefore, the following important corrections:

Three-hinged arches: Present theory, statically determinate system, no temperature strains.—Corrected theory, statically determinate system, temperature strains plus bending strains from three hinges.

Two-hinged arches: Present theory, singly indeterminate system, plus temperature strains.—Corrected theory, singly indeterminate

Mr. Lindenthal. system plus temperature strains, plus bending strains from two hinges.

Hingeless arches: Present theory, two-fold indeterminate system, plus temperature strains; requires, of course, no correction, inasmuch as there are no hinges.

Strain sheets prepared on the corrected theory, for the same rise of equilibrium curve for similar loads in all three types, are alone fairly comparable as to economy of metal. The three-hinged and two-hinged types require additions to the cross-sections of the members over those obtained on present theory. These additions cannot be ignored. When made, the two-hinged arch will in every case show great economy over the three-hinged arch.

The secondary stresses, which occur at the panel points from the elastic deformation of the frame, affect in practice only the details and strength of connections. They are not different from those in trusses, cantilevers, and other frames. They may be ignored in a comparison of strain sheets.

The objection to indeterminate structures cannot very well be based on the ground that the strains cannot be determined with the same exactness as in statically determinate structures, because such is not necessary, either for safety or economy.

The dangerous strains, their limits, and the conditions under which they occur, can always be ascertained with sufficient accuracy without the endless drudgery which, as Mr. Cilley shows, a detailed analysis for each member of a large structure would involve. There is a limit to the refinement of the engineer's calculations, beyond which nothing of practical value can be gained.

With the utmost accuracy of computation, in statically determined or undetermined structures, we will never be able to dispense with a three to ten-fold safety, as the case may be, to cover defects of manufacture, want of uniformity in the material, and the many petty indeterminate, as well as undetermined, stresses which are always present. The minimum and maximum limits of stresses ascertained, their frequency properly judged and provided for, we need have no anxiety for the safety of statically indeterminate structures.

No failures of bridges, caused by their indeterminateness, have yet occurred. Some, over 40 years old, are still carrying safely railroad loads, which, in the meantime, have largely increased. The greater diffusion of the shearing strains, particularly characteristic of indeterminate structures with multiple systems of web members, has prolonged their life. Repairs in a few cases are known to have been necessitated by poor details. Many statically determinate structures, built at the same time, and proportioned for the same unit stresses, have shown less resistance to wear. Some American railroads have already the third generation of metal bridges.

Mr. Lindenthal.

It is not irrelevant to the subject under discussion to refer here to the wide-spread bias against the very useful continuous girder, which is alleged not to be economical. This is true only when the chord members, subject to alternate stresses far within the elastic limit, are required to be largely increased in accordance with the discredited Launhardt formulas, or with other similar rules, unjustified by sound reasoning. We have no proof whatever of the strength of iron or steel being affected by alternating stresses within half the elastic limit. The increase of section, as required by bridge specifications in vogue, for members, subject to such strains, is a waste of money and material. The good results with the very economical, durable and rigid, continuous girders, proportioned and built before the present rules of dimensioning were known, ought to have weight with thinking engineers.

That the writer is not alone in his condemnation of the modern rules for alternate stresses in bridge construction, is gratifyingly evident from the report of the discussion at the International Railway Congress held in London in 1895.*

"But," quoteth the theoretician, "consider the dangerous strains in continuous girders, if a pier should settle, or when iron towers expand in height!" The answer is, that piers should be built so that they will not settle more than a certain allowable amount, and that the variations in the levels on iron towers can be allowed for readily in the dimensioning of the girders. This is a part of engineering science, and surely what the Chaldean and Roman bridge builders accomplished in the way of good foundations, the modern engineer should at least be able to equal. And when secure foundations cannot be had, neither continuous girders nor other forms of indeterminate structures should be chosen. It is the engineer's business to investigate and to discriminate.

The shaky and vibrating cantilever structure, so much affected as a great improvement in that respect, is by no means an adequate substitute for the rigid, compact and economical continuous girder, properly designed.

Another point deserves attention. The author seems to ascribe much importance to deflections as a means of judging the rigidity of a bridge. What is a rigid bridge? If we take rigidity in metallic bridges to mean absence of vibration, then deflections are a deceptive criterion.

The public considers a bridge which vibrates, as an inferior structure; and one which does not, as a superior one. This ought to be the guide, also, to the engineer. Sometimes, structures regarded as rigid have great deflections and little vibration, and *vice versa*, of which the following are a few instances:

The old Niagara single-track railroad suspension bridge (indeter-

* Congress Bulletin, page 79.

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minate) deflected ordinarily 10 ins., under a train, but a pedestrian on the roadway below would hardly notice it and would feel very little vibration. It was a more rigid structure than the double-track cantilever railroad bridge (determinate) located near by, which deflects ordinarily only 3 ins., but a pedestrian on it feels disagreeable sensations, and can hardly keep his feet when a train passes over the bridge.

The St. Louis arch bridge (indeterminate) vibrates very noticeably under every train and team, while the Merchant's Bridge (determinate) located near by is a fairly rigid truss structure.

The unsightly Market Street cantilever bridge (determinate) in Philadelphia is unpleasantly known for its spring-board motion, while the old cast-iron arc bridge (indeterminate), located below, at Chestnut Street, is rigid. One of the abutments of the latter bridge has yielded about 3 ins., but none of the calamities predicted in such cases for indeterminate arches have occurred. The behavior of three famous steel-arch bridges is also instructive, viz.: The Washington arch-bridge (indeterminate), in New York, has ribs with solid webs; its deflections from ordinary loads are nil, but its vibrations, even under a single-horse truck, are so noticeable that they have been the subject of (of course unfounded) anxious communications to newspapers. If it had three hinges instead of two, the vibrations would be worse. The Niagara Falls ribbed arch bridge (indeterminate), completed two years ago, is similarly complained of, while the spandrel-braced railroad-arch bridge (indeterminate) below is one of the most rigid metal bridges in existence.

Specially interesting, in that respect, are also four different types of large bridges at Buda Pest.

The railroad bridge with continuous lattice girders (indeterminate) is rigid under fast trains; the famous Buda Pest suspension bridge, with no stiffening to speak of (and therefore determinate), shows only little less vibration than the heavy Margarethen spandrel-braced, arch bridge (indeterminate), or the new cantilever bridge (determinate) near by.

Owing to the prejudice against indeterminate structures, most of the very high, iron-trestle viaducts in this country are subject to so much vibration, that trains must greatly reduce speed over them. With girders cut over every post, and provided with slide bearings, as demanded by the usual bridge specifications, sufficient rigidity cannot be obtained. The high viaducts, with continuous girders (indeterminate), in Europe, some of them over 40 years old, are, on the other hand, fairly rigid under fast trains.

Compare the Forth Bridge (indeterminate), proportioned for a live load of 4 000 lbs. per lineal foot, a riveted cantilever structure, in which 15% of the metal is in the strong lateral bracing, with the Brooklyn

Suspension Bridge (indeterminate), proportioned for only 2 000 lbs., having no lateral bracing to speak of, and a very defective stiffening system. The latter bridge is subject to much greater deflection than the Forth Bridge, but both bridges are equally free from noticeable vibration under the loads and at the speeds for which they are designed. Mr. Lindenthal.

The remarkable rigidity of even scantily stiffened suspension bridges and the reasons therefor have been discussed by the writer on former occasions. It is his opinion, that for long spans properly stiffened suspension bridges are the most rigid of any metal type.

Thus the study of existing bridges, of both the determinate and indeterminate kind, affords better instruction in rigidity than the mere comparison of deflections. Naturally, these ought to be always as small as possible.

The writer does not advocate indeterminate structures, but neither is he prejudiced against them. A decision as to their true economy and merits can be reached only from case to case. It seems to him that the sweeping conclusions of Mr. Cilley and of others against them are founded on incomplete investigations and are at variance with known facts.

Prof. C. W. RITTER* (by letter).—In his detailed comparison of statically determinate and indeterminate frameworks, the author endeavors to demonstrate the decided superiority of the determinate forms. He claims that the determinate structures are, theoretically, more economical than the indeterminate ones. He shows that in indeterminate structures, a slight inaccuracy in the length of a member or a slight change of temperature, can effect a considerable and even serious change in the stresses. He emphasizes the fact that discrepancies in the levels of the supports of continuous girders, or a slight yield of piers or abutments of two-hinged and hingeless arches, have considerable effect on the interior stresses. He also points out the laboriousness of the exact design of indeterminate structures. Mr. Ritter.

It will scarcely be necessary to concede that all these arguments are doubtless very reasonable, and the writer thinks it wise to emphasize now and then the advantages of determinate structures and to expound the uncertainties of the basis on which the design of indeterminate structures is founded.

In German literature, these questions have often been discussed and analyzed. A thorough investigation of the matter has not been without conclusive results. The use of the multiple intersection system has commonly been dropped; arches which, formerly, were all designed without hinges are now very often provided with two or three of them. Within the last ten years, even several stone arches have been built

* Professor of Civil Engineering, Federal Swiss Polytechnic, Zurich.

Mr. Ritter. with three hinges. The cantilever system has been frequently substituted for the system of continuous girders.

Although a noticeable change has taken place in the manner of constructing frameworks, European engineers have not, as yet, abandoned indeterminate forms entirely, and the writer dares to say they never will. Without disputing the advantages of the determinate forms, a thorough and careful examination of the question leads to the result that, considered from the practical standpoint, the indeterminate frameworks must frequently be declared preferable, at least for European practice.

First, it may easily be perceived that some of the above-mentioned disadvantages of the indeterminate structures are, in many cases, of little weight, and that some others adhere also to the determinate forms. If the foundations are absolutely solid, why should we not build a continuous girder, as well as a cantilever. The influence of varying temperature on two-hinged and hingeless arches decreases with increasing height for the same span, and, in case such arches have other advantages besides, we should not hesitate to prefer them to the three-hinged arch. Unequal warming of the members of an indeterminate framework exerts indeed a disadvantageous influence on the stresses, but the same may be said of the one-sided warming of the members of a determinate structure. In fact, we will never succeed in avoiding entirely the influence of varying temperature.

In regard to economy, the author asserts that the determinate structures need less material than the indeterminate ones. For this assertion he gives a plain and convincing mathematical proof. But let us be careful in applying this result to our practical constructions.

For instance, it can be shown that the framework designated *a*, Fig. 18, which is determinate, requires more material than the framework designated *b*, which is indeterminate. Assuming each panel load to be equal to 4, and that, by introducing initial stresses, the diagonals in each panel in form *b* are equally strained, the stresses indicated by the figures will result. Multiplying these stresses by the lengths of the respective members, we have:

Form *a*—Upper chord	$= 2\,(10 + 16 + 18)$	$= 88$	248
Lower chord	$= 2\,(10 + 16)$	$= 52$	
Posts	$= 10 + 6 + 4 + 6 + 10$	$= 36$	
Diagonals	$= 2\,(14.14 + 8.49 + 2.83)\,\sqrt{2}$	$= 72$	
Form *b*—Chords	$= 2\,(10 + 13 + 17 + 13 + 17)$	$=140$	232
Posts	$= 7 + 2 + 2 + 2 + 7$	$= 20$	
Diagonals	$= 2\,[14.14 + (2\times 4.24) + (2\times 1.41)]\,\sqrt{2}$	$= 72$	

Nevertheless, the assertion of the author is correct, inasmuch as there exists a third form *c*, included in *b*, which requires still less material, for we have: Mr. Ritter.

Form *c*—Upper chord	$= 2\,(10 + 12 + 16)$	$= 76$	224
Lower chord	$= 2\,(14 + 18)$	$= 64$	
Posts	$= 6 + 6$	$= 12$	
Diagonals	$= 2\,(14.14 + 5.66 + 2.83 + 2.83)\sqrt{2}$	$= 72$	

At the same time, we recognize that the inner posts have disappeared. As these posts, as a rule, are necessary for attaching floor-

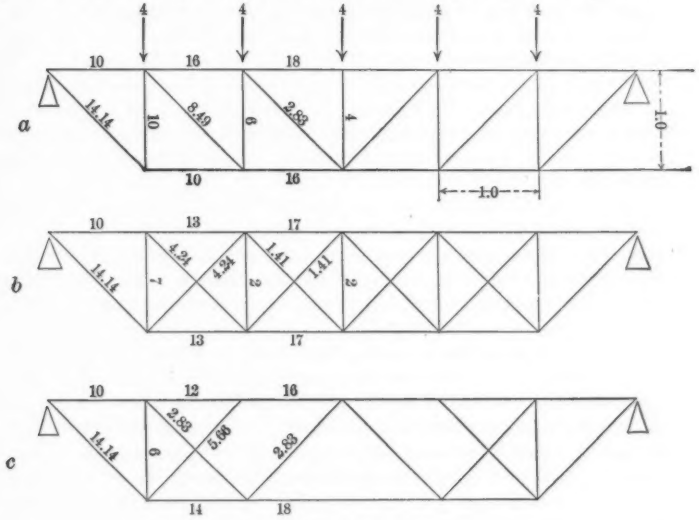

FIG. 18.

beams, the form *c* is in most cases practically useless, which proves that the advantage of the determinate frameworks, as far as the theoretically required amount of material is concerned, may, by accessory circumstances, easily be reversed into a disadvantage.

The author himself shows that for varying loads the above-mentioned rule loses its validity. There are still other facts which often prove the indeterminate structures to be more economical.

First, in using intersecting diagonals, we obtain the advantage that in these members the stresses are halved, and, in consequence, the struts can be attached directly to the chords; while, in the contrary case, we are obliged to use costly connection plates. Again, by intersecting the ties with the struts, we shorten the distance of supports

Mr. Ritter. of the latter. Further, it is a common experience, that the stresses in the various members of an indeterminate framework are of greater uniformity than those of a determinate one, a fact which naturally leads to some economy. Finally, for cantilevers and three-hinged arches, the cost of the structure is increased by that for the hinges, and it is not unusual that this additional cost exceeds all economy secured otherwise through statical determination. It may be remarked also that the connections of the lateral system, on account of its continuity past the hinges, very often lead to clumsy details and difficulty in construction.

After all, it will be difficult or even impossible to predict which of the two systems will prove cheaper in the end, without computing and designing, in each special case, both of them. The greater labor involved in computing statically indeterminate structures can rarely be a reason for abandoning them. The methods for their computation have been perfected to such a degree that the time required for it alone is hardly of any moment in comparison with that required for the entire design.

One of the chief objections, emphasized by the author, to statically indeterminate structures is the possible discrepancies in the length of members. This objection is an important one for the United States of North America, where pin-connected structures are generally built. It is of less importance in Europe, where riveted connections are used more generally. If the length of any member of an indeterminate, pin-connected framework is inaccurate, the defect, as a rule, cannot be corrected easily, while in riveted connections such deficiencies are compensated by the red-hot rivets which will fill out the holes even if these, within certain limits, do not match. European specifications, not unlike American ones, strictly provide that if, in connecting parts or members, the rivet holes fit badly, the holes have to be enlarged and larger rivets have to be used. Stretching the bars, in order to match the rivet holes, is strictly forbidden. If in spite of these precautions a slight discrepancy remains, it is, as mentioned above, compensated by the rivets themselves, which, if properly driven, will fill out the holes perfectly. It seems to the writer that for this reason the inducement for avoiding statically indeterminate structures exists far less for rigidly rivet-connected than for pin-connected structures.

Regarding stiffness, the author gives us an interesting comparison between the deflection of a two-hinged and a three-hinged arch. He finds that the difference is very insignificant. It seems to the writer that this result, for the example in question, has to be attributed to the fact that the theoretical height of the two-hinged arch is actually less than that of the three-hinged one. In general, structures with hinges (arches or cantilevers) are less stiff than those without hinges. For instance, the fixed span of a cantilever deflects exactly as much as a

single-span girder, while the deflection of a continuous girder of the same span is from 30 to 50% less. Let it be remembered that in railway bridges such hinges often produce shocks, which benefit neither the bridge nor the rolling stock. Mr. Ritter.

In reference to the stiffness of bridges, we should not only consider the elastic deflections under varying loads, but also the vibrations of the structure in a vertical and lateral sense. It can scarcely be contended that, in this respect, statically indeterminate structures are safer, although, as the author says, a strict proof for this assertion can scarcely be given. Now, is not the extensive adherence of American engineers to stiff joints in upper chords, as well as the rigid attachment of floor-beams, stringers, and lateral, portal and sway-bracing, a proof of their sound constructing sense, although all these construction details clearly involve statical indetermination? The author, it is true, refers principally to vertically loaded frameworks, but what is sound for floor beams, stringers and bracing, is not likely to be wrong for frameworks throughout.

The author should be thanked for his intelligent investigations and his manifold suggestions. In any case, it will be unreasonable to build an indeterminate framework where a determinate one is just as advantageous, and the author's endeavor to substitute determinate for indeterminate forms is praiseworthy. But on the other hand, it is the writer's opinion that indeterminate structures are in many cases preferable, and that we will never be able to do without them entirely. Their frequently greater economy and their greater stiffness, as well as their adaptability to various circumstances, are such important advantages, that it would not be wise to throw them overboard on mere principle. On the contrary, it should be regarded as the task of engineering science to study the qualities, the peculiarities and especially the deficiencies of indeterminate structures, to search for necessary remedies, and to simplify further methods of computation.

Prof. W. Dietz* (by letter†).—The writer was a pupil of Head-Master H. Gerber, who, by the introduction of continuous girders, such as cantilevers and those simply supported at the piers, has gained great distinction in the art of bridge building, and who, as is perhaps not generally known, by the construction of the street bridges over the Main, near Hassfurt,‡ furnished the prototype for the truss system of the Firth of Forth Bridge. The writer, therefore, has had the opportunity of becoming intimately acquainted, both theoretically and constructively, with the greatest variety of statically determined truss systems. It follows, therefore, that he has retained a certain preference for such systems, with their strains so clearly and quickly Mr. Dietz.

* Technical Highschool, Munich.

† Translated.

‡ *Zeitschrift für Architektur und Ingenieurwesen*, 1898, p. 563, Fig. 1.

Mr. Dietz. obtained. Yet his vocation makes it necessary for him also to investigate, thoroughly, both theoretically and practically, statically indeterminate systems of the most different kinds, and his experience gained thereby, together with the indisputable advantages of undetermined systems under certain conditions, compel him not to withhold due acknowledgment thereof, and to protest against unfounded attacks upon the existence of these trusses.

The writer, with a large amount of material at his disposal, must characterize the author's closing sentence "that the use of indeterminate framework is certainly not advantageous from an economic standpoint, nor apparently from any other standpoint," as not in accordance with facts, and hardly capable of demonstration.

The matter is not as simple as the author represents it to be. As is well known, it is not only the length of span between supports that influences the volume of iron in bridge trusses, but also the greater or smaller number of panels into which the span is divided, the length and form of cross-section of the compression members; the kind of joints, and the manner in which they are made, as also the riveting of adjoining parts; and, finally, to a very great extent, the method used in dimensioning. As regards the latter, the method by Gerber, as specified in Bavaria, which takes into account the impact of variable loading, gives considerably different cross-sections in individual members from those determined by the widely used Launhardt-Weyrauch formula, which does not consider the effect of impact.

In order, therefore, to make a professional comparison, in regard to the amount of iron needed, between several different truss systems, it is necessary to take account of all the foregoing factors, and to observe the condition that all the trusses are dimensioned by the same method.

Most of these factors are taken account of by introducing the so-called construction coefficient with which the theoretical volume of the different members must be multiplied in order to obtain the practical volume. The determination of these coefficients is, however, a work of considerable scope, and leads to results of practical value only when based upon a large number of data, which the bridge companies, for reasons which are obvious, do not give out generally.

Further, it should not be overlooked that the superstructure of a bridge forms only a part thereof, and that not only the metal construction, but also the abutments and piers, the method of erection and the maintenance of the structure, should be taken into consideration, so that in special cases it is quite possible that a considerable increase in the cost of the iron structure may result in a saving on the final cost of the whole bridge. The ultimate object of all our technical information and knowledge is to arrive at the lowest possible total cost, providing at the same time for an even strength of all the parts.

Mr. Dietz.

There is ample proof to show how often the effort is made in Germany, even in extraordinary cases, to use statically determined truss systems, as, for instance, in the Neckar Bridge, in Mannheim* and the Donau Bridge in Budapest.† If, however, in a large number of structures‡, built of late, indeterminate frameworks have been used in the main truss systems, the reason for this should not be sought in a one-sided preference for difficult theoretical problems, but in the fact that after a careful consideration of all circumstances in each case, these systems were found to be the most advantageous.

As a particularly instructive example of this class, the Müngsten Viaduct§ can be mentioned. In this case the most thorough comparative studies regarding the effects of brakes, and the matter of erection, led finally to the selection of a combination of trestle bridge with a statically indeterminate arch with fixed ends.

It should be noted, in this connection, that very interesting experiments were made upon this bridge, and that they showed a very satisfactory agreement between the theoretical and the actual strains in the truss members.

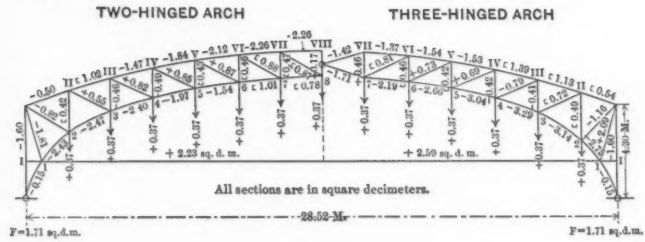

FIG. 19.

As is well known, the Germans are much interested in the thorough investigation of the so-called secondary strains‖ which are found in every truss of whatever system, arising either from the inflexible, riveted joints, or from the frictional resistance of the pins, which, even in the best constructed, pin-connected bridges must be overcome before an actual turning of the individual members about the pin can take place. From this point of view the two-hinged arch, for instance, is, when properly designed, considerably superior to the ordinary truss bridge resting on two supports; and this is an additional factor which in connection with a pleasing outer appearance,

* *Zeitschrift des Vereins Deutscher Ingenieure*, 1891, p. 89, Fig. 12, and *Zeitschrift für Architektur und Ingenieurwesen*, 1898, Plate xiv, Fig. 11.

† *Zeitschrift des Vereins Deutscher Ingenieure*, 1894. International Competition, p. 1238, Fig. 72.

‡ *Zeitschrift für Architekture und Ingenieurwesen*, 1898, p. 561.

§ *Zeitschrift des Vereins Deutscher Ingenieure.* "The Müngsten Viaduct." p. 1321.

‖ Engesser: "Supplementary Forces and Secondary Strains in Iron Truss-Bridges," 1892 and 1893.

Mr. Dietz. which the Germans happily are appreciating more and more, has influence in determining the choice of a truss system.

In the preceding, the writer has outlined the different elements which should be considered in judging the merits of different truss systems; he would now, in conclusion, prove by figures how little permissible it is to draw general conclusions as to the economical superiority of the three-hinged over the two-hinged arch, even in purely theoretical respects, from the author's comparisons between quite small arches of definite shape.

The diagram, Fig. 19, illustrates the very carefully made comparisons between a three-hinged and a two-hinged arch with elastic tie rod for the Hacker Bridge in Munich.* In the right half the theoretically determined cross-sections for a three-hinged arch are given, and in the left half those for a two-hinged arch, in square decimeters. Pure tensile strains are indicated by +; pure compressive strains by —, and members subject to both kinds of strain by [.

The result of this comparison is very unfavorable to the three-hinged arch, even when only the theoretically determined volumes are considered, inasmuch as it requires 11.3% more material than the two-hinged arch, and this difference becomes still greater when the additional material required for the center hinge is taken into account. A still greater disadvantage for the three-hinged arch appears when its deflection is considered, as it amounts to 30.5% more than that of the two-hinged; it was calculated carefully for both arches, according to the well-known method by Mohr, taking into account the change of length in every member.

The writer maintains that the question, whether under all circumstances the statically determinate framework is superior to the indeterminate, cannot at all be answered on the basis of a one-sided comparison, considering only the theoretically required amount of material. It can only be determined in each case by weighing carefully the influences of all the factors, and requires, not only thorough theoretical knowledge, but also, a large amount of comparative material and experience.

Mr. Sohn. Prof. JOSEPH SOHN (by letter).—The exact design of statically indeterminate frameworks, as explained in the paper, is based essentially on the idea that the stresses of the superfluous bars under a certain loading are not computed but assumed. In the case of a single given loading (constant loading), the author assumes arbitrarily the stresses in the superfluous bars, deduces thence and from the loading the stresses in the remaining (necessary) bars and designs on this basis the cross-sections of the various members. The equations of elasticity determine then the differences between the figure length and the actual un-

* *Zeitschrift des Vereins Deutscher Ingenieure*, 1893. "Construction of the Hacker Bridge" p. 1441.

strained length of each superfluous bar. Thereby, the construction is designed, and in practice it only remains to give to the bars the corresponding length. The more nearly this condition is attained the more nearly the actual stresses approach the designed. Mr. Sohn.

If a statically indeterminate framework is subject to the several different loadings (variable loading), the author assumes the stresses in the superfluous bars under one of the given loadings or, more generally, he sets an equal number of further conditions; in the equations of elasticity, each relating to the difference between the stresses in the bars under two of the given loadings, and therefore not containing the differences between the figure and actual unstrained length of the superfluous bars, the author expresses the section areas of every bar by the corresponding stress under the most unfavorable loading; the equations of elasticity determine, then, the stresses in the superfluous bars under each of the given loadings; thence result the stresses in the necessary bars, and it remains to compute the differences between the figure length and actual unstrained length of the superfluous bars.

The case of a constant loading offers, in designing, no difficulty; with variable loading, on the contrary, the design, even under the simplest conditions, is exceedingly complicated and laborious. The author himself accents it expressly and adds force principally upon the proof that statically indeterminate systems are in all respects (economy, stiffness, safety) inferior to the determinate constructions, and that only these last systems may be recommended in practice.

Although there are cases where it will not do to exclude statically indeterminate forms, the writer agrees with the author that statically determinate systems generally are to be preferred, particularly for the reason that slight imperfections of the actual construction, unavoidable in practice, do not influence essentially the stresses in the bars. In statically indeterminate constructions that influence is, on the contrary, very considerable; wherefore important divergences between the computed and actual intensities of stress are possible and probable. A simple example, given by the author, shows evidently how much the intensity of stress can be altered by the incorrect length of a bar.

G. Jung,* Esq. (by letter†).—The writer has read Mr. Cilley's paper on indeterminate frameworks with much interest and takes pleasure in finding that he entirely agrees with him. Mr. Jung.

For a long time the writer has maintained that statically determinate frameworks are preferable to indeterminate ones. His reasons for this opinion being about the same as those which the author has so well set forth, he considers it useless to repeat them here.

On the other hand, he has for many years, in his course on graphical

* Associate Manager of the *Annali di Matematica*, Milan, Italy.

† Translated.

Mr. Jung. statics at the Superior Technical Institute of Milan, called the attention of students to the paper of Maurice Levy (mentioned also by Mr. Cilley), and has invited them to study the argument treated therein. With considerations of a general character, analogous to those given on pages 570 and 571 of the paper, the writer emphasizes the simplicity of the calculation and of the hypotheses required by statically determinate frameworks as compared with the complicated calculations and the necessity of further hypotheses, required by indeterminate frameworks, and concludes with pointing out the superiority of the former structures over the latter, at least from the theoretical and purely statical (and economical) point of view.

As for the dynamical phenomena, the writer does not know of any; but the author's observations on page 361 seem to be just. The writer does not know, either, what the good reasons are which could make one prefer indeterminate rather than statically determinate frameworks, which latter have in their favor, besides, the practice and the large experience of American engineers.

Mr. Cilley. FRANK H. CILLEY, S. B. (by letter).—The writer believes that Mr. Goldmark's objections rest, in part, on a misapprehension. The use of methods of calculation based on the principle of virtual "Velocities" or "Displacements" or "Work," as it is variously termed, or which result from the principle of least work, which is simply an integral form of the preceding, is the simplest as well as the most correct means of determining changes in stress and strain in indeterminate structures. Where such structures exist, and, therefore, must be examined, the writer most strongly approves the use of these methods. And in designing, if engineers will use indeterminate forms, these methods are still to be preferred. But in this case the present tentative method of application must be used, as the direct and exact method outlined by the writer is too difficult to be practicable. The writer's point is that while the methods just referred to are the best known, still, being indirect and tentative, and embodying no principle that guides to good rather than bad proportions, even their use in designing is necessarily unsatisfactory, and, consequently, designs depending on their use will be correspondingly defective. The writer's objections are directed rather to the unnecessary use of systems of construction involving these uncertain, wearisome and unsatisfactory calculations than to the methods of making these calculations.

Regarding the alleged tendency to multiple systems of webbing, the writer would refer Mr. Goldmark to Professor Ritter's statement in his contribution to this discussion, that "the use of the multiple intersection system has commonly been dropped" as a result of German investigations of the subject.

Mr. Goldmark's references to the bibliography of the subject are

valuable. The writer personally prefers the use of the principle of "Virtual Velocities" (displacements or work), so clearly set forth in Professor Swain's article, "On the Application of the Principle of Virtual Velocities to the Determination of the Deflection and Stresses in Frames," elsewhere referred to, to the use of the integral principle of "Least Work," of which an excellent exposition by Professor William Cain will be found in the *Transactions** of this Society. The former principle, while leading to precisely the same equations as the latter, is much more objective, and, in the writer's opinion, is less likely to introduce errors. Mr. Cilley.

Mr. Lindenthal's objections raise many most interesting points which the writer only fears his present very limited time will not permit him to discuss with the thoroughness which is desirable. He trusts that a further opportunity will permit of his correcting present shortcomings, and that brevity will be pardoned in this instance.

One of Mr. Lindenthal's first objections is to the comparison of a given indeterminate framework with a determinate framework of included figure, a procedure which he characterizes as unfair. In this the writer will agree with him, except that the unfairness is not to the indeterminate, but to the determinate form. For the indeterminate form is any given form, and, certainly, it would be fair in theory and always permissible in practice to use in its place another framework of included figure. It rises no higher, it descends no lower, it extends in nowise beyond the bounds of the given framework, and offers at least as great clearances in all directions. What just objection, then, can be made to its substitution for the given framework? And if it is true that this leads to a greater "effective" height or depth in the substituted framework, that is only one of the legitimate advantages of the substitution. It is in part because of such advantages that the determinate forms are gainers over the indeterminate. But the comparison, even so, is unfair to the determinate framework, for all the proportions of the indeterminate framework may be those which most favor it, while the determinate framework, whose figure is formed simply by dropping out certain lines of the figure of the given indeterminate framework, without any alteration in proportions, may not, and very likely will not, be the best determinate framework whose figure simply does not exceed the bounds of the figure of the given indeterminate framework. This latter would certainly be a fairer basis, but even that would not be wholly fair. Usually, bounds in many places exceeding those of the given indeterminate framework would be equally permissible. It would only be fair to permit the search for a superior determinate framework to extend to any within these bounds. It must be remembered that the indeterminate framework is supposed to be given and the question raised is: Can a determin-

* *Transactions*, Am. Soc. C. E., Vol. xxiv, p. 265.

Mr. Cilley. ate framework be found which would equally well satisfy the actual limiting conditions, and which would advantageously replace the given framework? Could this be shown in general to have an affirmative answer, the superiority of determinate over indeterminate construction would thereby be demonstrated conclusively.

But it is not possible to thus broaden the limits in a general discussion. We must remain within limits which are beyond question, and those surely are the boundaries of the given indeterminate framework. The writer has even narrowed these limits still further, and required the bars of the determinate framework to occupy no other positions than those of corresponding bars in the given indeterminate framework. As a result, while the demonstration that there exists a superior determinate framework under these conditions is absolutely conclusive, the demonstration of the contrary would still leave open the possibility of a reversal through fairer conditions.

Mr. Lindenthal's next point is an objection directed to the very considerable proportion of depth to span in the arch of the second illustration. It is one-eighth, whereas in practice one-sixteenth is about the maximum and one thirty-second more frequent. Now, this arch was proportioned as a two-hinged and not as a three-hinged arch. The small number of its members (required to reduce the work of an "exact design") alone is responsible for this considerable depth. The three-hinged design obtained by dropping out the middle bar of the two-hinged design is far from being of a superior character. For the limited number of members used, the three-hinged design is by no means favored in its proportions. If it gains decidedly in effective arch rise at the center through this great depth of the two-hinged arch, it may be contended, on the other hand, that the great depth of the two-hinged arch at the middle should secure it great rigidity. We should expect that where the characteristic difference of two types was most marked, there we would find the most marked expression of the superiority of one or the other. And surely it is the absence of all stiffness at the center of the three-hinged arch, and the considerable stiffness at the center of the two-hinged arch, which are the most characteristic contrasting features of the two. If, where these are most marked, we find economic and other differences small, how much less will they become as these differences in character become less marked. It must be remembered that the writer is contending that decided advantages in economy, stiffness and safety are not obtainable through the use of indeterminate frameworks, and, therefore, that other considerations favoring determinate construction decidedly outweigh any small advantages in these directions which indeterminate structures may occasionally possess. It is, therefore, a very important point to have shown, as in the illustration to the paper, that even in an extreme case, where large divergence was to

be expected, actually, it was small. Therein, in part, is the justification for the statement that in actual and usual cases the differences will be much smaller still. Mr. Cilley.

We now come to another point, and this is a very important one, in the comparison of determinate and indeterminate forms—the question of flexibility. If a structure be very flexible, that is, permit of relatively very considerable changes of form, if it is an indeterminate structure, it is evident that the importance of adverse factors, such as temperature changes, yielding supports and inaccurate construction, will be greatly lessened. That is to say, in slender structures the defects of indetermination tend to become negligible. Now, an arch with a rib depth less than one thirty-second of the span certainly is a slender and relatively flexible structure, and, if arches are usually of such proportions as Mr. Lindenthal states, then they are usually in the category of structures in which the evil consequences of indetermination are minimized. Ordinary, stiffened suspension bridges are also in this category.

Now, these flexible types really form a class by themselves. Their distortions under load are so considerable that the ordinary theory of rigid structures does not closely apply to them. The terms "statically determinate" and "indeterminate" cease to have their old meanings, and consideration of elastic displacements becomes necessary in all cases. It is true that such consideration is not ordinarily given, but that is chiefly because of our ignorance and its difficulty—not because it is not needed.

The writer's study can make no pretense of including such structures. Based on the application of the principle of virtual displacements, it certainly does not apply when the actual displacements result in very sensible directional changes. And it will be well, at this point, to eliminate from this discussion the discussion of the comparative merits of determination and indetermination in such structures. It is an interesting subject, but occupies a field by itself. Here, we are only concerned with the location of its border line. The writer ventures the temporary suggestion that structures whose changes in form under load result in a change of more than 10% from the stresses which would result if the changes in form were insignificant, be considered as of the class of flexible structures here excluded.

Limiting ourselves, then, to the consideration of non-flexible structures, the writer's statement of the great importance of temperature stresses in indeterminate structures is well founded, and his illustration with the deep arch a proper example. The suspicion is probably justified that, in indeterminate arches in which the temperature stresses have been found to be very small, the distortions are actually very considerable, and the calculated stresses under load, obtained in the usual ways, far from expressing the actual stresses.

Mr. Cilley. Mr. Lindenthal tells us that:

"Comparisons of carefully worked out designs of different arch types, made by him and others, have convinced him that, other things being equal, the indeterminate types are invariably more economical."

Without questioning his sincerity in this statement, it may fairly be asked that he shall produce the data in order that judgment may be passed on the fairness of the comparisons. The writer's experience in these matters is, that really fair comparisons are practically unknown, not because of prejudice on the part of those making the comparisons, but because of the unconscious introduction of unreasonable limitations. Thus, if a three-hinged arch is to be designed for comparison with a two-hinged arch, the middle hinge is usually placed midway between the top and bottom chords, or at the bottom chord, in spite of the obvious fact that the horizonal thrust is diminished by putting the hinge as high as possible; that is to say, in the top chord. Or, if it is a comparison of the three-hinged arch with a hingeless arch, two of the hinges are always placed at the abutments, in spite of the fact that it may be decidedly more advantageous to put them farther out, and that it is legitimate to do this and subject the abutments to a bending moment which they would have to resist in any event with the hingeless arch.

Another form of unfair discrimination is that mentioned by Professor Jacoby,* where he speaks of the designing of a three-hinged arch to compare with a given two-hinged arch, and giving the sections of the lower chord of the former such proportions that, in spite of its being heavier, its radius of gyration averaged 4% less than that of the sections of the lower chord of the two-hinged arch. In spite of this fact, however, in this case the three-hinged design proved slightly the more economical. The designing of the cross-sections is a most important factor in fair comparisons, and it alone, by its variation, may easily throw the balance in favor of one type or the other in the comparison of actual designs. This fact is one of those which render the comparison of actual designs so unreliable a basis for determining the relative economy of types.

We now come to a point on which Mr. Lindenthal correctly insists, but, the writer fears, without due regard to proportion—the fact that changes in temperature modify the stresses, even in many determinate frameworks, contrary to the prevalent idea. But Mr. Lindenthal goes so far as to state that:

"Three-hinged arches are not only not free from temperature strains, but for certain conditions such strains are just as large as in arches of the two-hinged type. * * * The supposed advantage in that respect of the three-hinged over the two-hinged arch is illusory."

While the changes in stress resulting from temperature (which are not precisely temperature stresses) are facts and should be considered,

* *Transactions*, Am. Soc. C. E., Vol. xliii, p. 31.

the writer feels that Mr. Lindenthal's quantitative statements should be accompanied by the data on which they are based so that we may see for what "certain" conditions these stresses accompanying temperature changes are so considerable. Mr. Cilley.

If we look into these so-called temperature stresses of Mr. Lindenthal a little closer we shall note that they are simply a special case of the modification of the stresses in a structure resulting from changes in its form, however brought about. The loadings bring about such changes in form, and, as previously noted, to such an extent in the more flexible structures that the ordinary theory which neglects such changes is seriously in error. Now the fact is simply this, that structures whose figures are altered seriously by temperature changes are likely to have their figures altered even more seriously by their loading; that is to say, are to be classed among the flexible structures mentioned previously. While calling attention to these temperature changes Mr. Lindenthal is neglecting other, even more important, changes which occur in these cases.

And another fact is to be observed: These neglected changes, whether from temperature or load, occur in the indeterminate frameworks as well as in the determinate, and add their effects to one as well as to the other; so that Mr. Lindenthal's neglected factor is one that must be applied, both to the indeterminate and the determinate frameworks. Very flat arches, whether three-hinged, two-hinged or hingeless, suffer serious alteration in their stresses from those ordinarily determined, through their flattening very sensibly further, whether from fall of temperature or from the application of the loads. It is a mistake to suppose that the present usual theory of indeterminate structures takes this flattening into account, whether in connection with temperature or load stresses. Actually, it neglects it as completely as does the ordinary theory of determinate structures. To see this and to enable us to form a better idea of these alterations of stress, we must develop their theory.

Theory of Changes of Stress in Frameworks Due to Changes in Form.

Suppose we denote by P the loads, by S the stresses, by R the reactions, and by α the angles of the bars, loads and reactions with (say) the horizontal, confining ourselves here, for simplicity, to plane frameworks. Then, as we know, on the ordinary supposition of no appreciable change in form of the framework under load for each joint we have equations of the form:

$$\left.\begin{aligned} \Sigma P \cos.\ \alpha + \Sigma R \cos.\ \alpha + \Sigma S \cos.\ \alpha = 0 \\ \Sigma P \sin.\ \alpha + \Sigma R \sin.\ \alpha + \Sigma S \sin.\ \alpha = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots (A)$$

But if the framework actually does change in form and the α

Mr. Cilley. become $\alpha + \Delta_\alpha$, the S become $S + \Delta_S$, the R become $R + \Delta_R$, and the P become $P + \Delta_P$, then our joint equations become:

$$\left.\begin{aligned} \Sigma\,(P + \Delta_P)\cos.\,(\alpha + \Delta_\alpha) + \Sigma\,(R + \Delta_R)\cos.\,(\alpha + \Delta_\alpha) + \\ \Sigma\,(S + \Delta_S)\cos.\,(\alpha + \Delta_\alpha) = 0. \\ \Sigma\,(P + \Delta_P)\sin.\,(\alpha + \Delta_\alpha) + \Sigma\,(R + \Delta_R)\sin.\,(\alpha + \Delta_\alpha) + \\ \Sigma\,(S + \Delta_S)\sin.\,(\alpha + \Delta_\alpha) = 0. \end{aligned}\right\} \ldots.(B)$$

Each of these terms, as $(S + \Delta_S)\cos.\,(\alpha + \Delta_\alpha)$ may be expanded as $(S + \Delta_S)\cos.\,(\alpha + \Delta_\alpha) = S\cos.\,\alpha\cos.\,\Delta_\alpha - S\sin.\,\alpha\sin.\,\Delta_\alpha + \Delta_S\cos.\,\alpha\cos.\,\Delta_\alpha - \Delta_S\sin.\,\alpha\sin.\,\Delta_\alpha$.

Now suppose the Δ all relatively small, then we could write $(S + \Delta_S)\cos.\,(\alpha + \Delta_\alpha) = S\cos.\,\alpha - \Delta_\alpha S\sin.\,\alpha + \Delta_S\cos.\,\alpha$, and similarly with the other terms of (B), and taking Equations (A) into consideration we thus obtain from Equations (B) the difference equations for each joint:

$$\left.\begin{aligned} \Sigma\,\Delta_P\cos.\,\alpha + \Sigma\,\Delta_R\cos.\,\alpha + \Sigma\,\Delta_S\cos.\,\alpha - \Sigma\,\Delta_\alpha P\sin.\,\alpha \\ - \Sigma\,\Delta_\alpha R\sin.\,\alpha - \Sigma\,\Delta_\alpha S\sin.\,\alpha = 0 \\ \Sigma\,\Delta_P\sin.\,\alpha + \Sigma\,\Delta_R\sin.\,\alpha + \Sigma\,\Delta_S\sin.\,\alpha + \Sigma\,\Delta_\alpha P\cos.\,\alpha \\ + \Sigma\,\Delta_\alpha R\cos.\,\alpha + \Sigma\,\Delta_\alpha S\cos.\,\alpha = 0 \end{aligned}\right\} \ldots(C)$$

If the framework be determinate, then, since the P and α are known, the R and S determinable from the Equations (A) or their equivalents, the Δ_α (except for the reactions) very approximately determinable from the elastic displacements of the joints, for the given materials and sections, due to the stresses S, temperature and other causes, and the Δ_P either zero or definitely related to these displacements, Equations (C) suffice to determine our remaining unknowns, that is to say, the quantities Δ_R, Δ_{α_R}, and Δ_S.

The above result suffices for the solution of the problem, but it is capable of an exceedingly simple and objective interpretation which extends at once to the calculation of these changes in stress all the methods and resources, graphical and analytical, already developed for the calculation of the ordinary stresses. Consider the following:

If we inspect Equations (C) and compare them with Equations (A) we perceive that they are simply the equations of equilibrium at the joints of the given framework subject to the loads formed by taking the changes in the original loads, together with the further loads $\Delta_\alpha S$ at right angles to the corresponding bars. The resulting stresses in the framework under these loads are the desired changes in stress. So we have only to find this differential loading, which is readily done when the displacements of the joints have been determined (as by a Williot

diagram), and then we may proceed with the solution of the stresses by any of the usual methods, graphical or analytical, obtaining thus the change in stress resulting from change in form of the framework, in the case of any determinate framework. Mr. Cilley.

Applying the foregoing to the three-hinged arch of the second illustration of the paper, to determine the changes in stress due to change in figure resulting from a fall in temperature, we have the following: For a fall in temperature of 100° Fahr., if the arch did not alter its form, we would have a reduction of span of 0.4 × 0.065 ft. (see page 405) and a consequent descent of the middle hinge of $\frac{12}{40}$ of this, or 0.12 × 0.065 ft. But we suppose the span actually remains constant, and therefore the middle hinge must sink further, $0.2 \times 0.065 \times \frac{20}{12} = 0.333 \times 0.065$ ft. The total sinking of the middle joint will therefore be $(0.333 + 0.120) \times 0.065$ ft. = 0.030 ft., nearly. That is, all bars of the left half will rotate through the common angle,

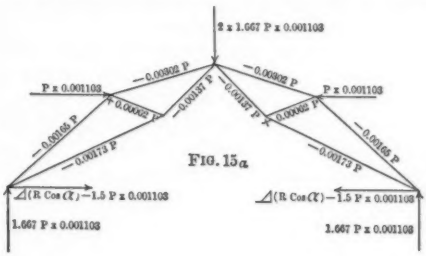

FIG. 15a

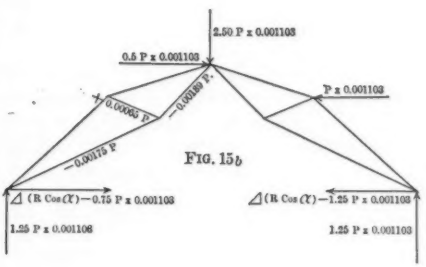

FIG. 15b

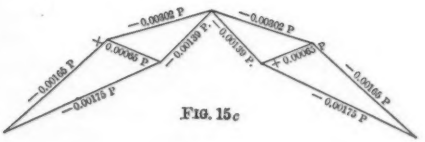

FIG. 15c

$$\text{tan.}^{-1}\frac{0.030 \times 20}{20^2 + 12^2} = \text{tan.}^{-1}\frac{0.6}{544} = \text{tan.}^{-1}\,0.001103$$ (about $3\frac{3}{4}$ minutes), in a right-handed direction, and all bars of the right half will rotate through the same angle in a left-handed direction.

Consider, first, the consequent changes in the stresses under a full loading of P on each joint (loading L, C, R).

We have in Fig. 15*a* the differential loading for this case and the resulting stresses which are the changes in stress sought. The value

Mr. Cilley. of $\Delta\,(R \cos.\ \alpha)$, or the change in the horizontal thrust, not given in Fig. 15*a*, is 0.00444 *P*, which permits of ready verification. Next consider the loading *C R* (*P* on center and right joints) under which bars *b*, *c* and *e* (see Fig. 2) have their greatest stresses.

We have, in Fig. 15*b*, the corresponding differential loading and the resulting stresses in the bars *b*, *c* and *e*, which are the changes in stress in these bars sought. The change in the horizontal component of the reaction in this case is found to be $\Delta\,(R \cos.\ \alpha) = 0.003125\ P$, which may easily be verified otherwise.

Putting down the changes in stress in the bars *a* and *d* under the loading *L C R*, and in *b*, *c* and *e* under the loading *C R* (and *L C*) we have, in Fig. 15c, the additions to the maximum stresses to be made on account of a change in temperature of 100° Fahr. Compare this with Fig. 15. Taking a change in temperature of only 50° Fahr. (that is, a range of temperature of 100° Fahr.), as in the case of the two-hinged arch, we find our additions are (compare Fig. 10) 0.09% in bar *e*, as against 22% for the two-hinged arch; 0.09% in bar *c* as against 20%; 0.09% in bar *b* as against 10%; 0.09% in bar *a* as against 11%; 0.035% in bar *d* as against 6%, to say nothing of no stress here to compare with the 45% temperature stress in bar *z* of the two-hinged arch.

This comparison, showing the stresses in the case of the three-hinged arch resulting from temperature changes to be only about one-two hundredth of the ordinary temperature stresses of the two-hinged arch, will be found a sufficient answer to Mr. Lindenthal's statement:

"If the author will trace the strains, due to the falling and rising of the center hinge from changes of temperature, he will find them by no means of negligible smallness, even for the large ratio of rise to span in his inconclusive figure."

But this is not all. The "temperature" (?) stresses we have just traced out in the three-hinged arch also occur in the two-hinged arch, in addition to those already calculated. Let us now develop the theory of these changes.

The theory of the changes of stress in indeterminate frameworks due to changes in form is not quite so simple as that for the determinate frameworks just set forth, but is based on the latter. At a first glance it would appear that one might simply find the differential loading for the indeterminate framework, and thence obtain the changes in stress sought simply as the stresses in the given indeterminate framework under this differential loading, on the supposition of no stress under no loading. But this is not quite correct as the following analysis will show.

We have seen (page 366), that the difference of the figure length L_i from the actual length l_i of a superfluous bar is expressed by the equation

$$\sum_{f=1}^{f=m} \frac{S_f'^{(i)}\, S_f\, l_f}{E_f\, A_f} = L_i - l_i \qquad (D)$$

provided only negligible changes in direction of the bars occur. If, as a result of the actual changes in direction of the bars, the stresses actually become $S_f + \Delta_{S_f}$ and the coefficients $S_f'^{(i)}$ become $S_f'^{(i)} + \Delta_{S'_f{}^{(i)}}$ then we must have Mr. Cilley.

$$\sum_{f=1}^{f=m} \frac{(S_f'^{(i)} + \tfrac{1}{2}\,\Delta_{S'_f{}^{(i)}})\,(S_f + \Delta_{S_f})\,l_f}{E_f\,A_f} = L_i - l_i \ldots\ldots\ldots (E)$$

And supposing, as before, that the Δ are small, we readily derive the simple difference equations

$$\sum_{f=1}^{f=m} \frac{S_f'^{(i)}\,\Delta_{S_f}\,l_f}{E_f\,A_f} + \frac{1}{2}\sum_{f=1}^{f=m} \frac{\Delta_{S'_f{}^{(i)}}\,S_f\,l_f}{E_f\,A_f} = 0 \ldots\ldots\ldots\ldots (F)$$

These Equations (F) together with Equation (C) actually determine the changes in stress, Δ_{S_f}. The second term on the left-hand side of Equations (F), which would have to be zero if the Δ_{S_f} were simply the stresses for the differential loading on the supposition of no stress under no load, shows the error that supposition would involve.

Instead of the purely analytic procedure of combining Equations (F) and (C), the following procedure is likely to prove preferable.

The stresses in tne non-superfluous bars are expressed in terms of those in the superfluous bars by the equations (see p. 371).

$$S_f = S_{f\,o} + \sum_{i=h}^{i=m} S_i\,S_f'^{(i)} \ldots\ldots\ldots\ldots\ldots\ldots (G)$$

on the supposition of no change in form. Actually, we must have

$$S_f + \Delta_{S_f} = S_{f\,o} + \Delta_{S_{f\,o}} + \sum_{i=h}^{i=m} (S_i + \Delta_S)\,(S'_f{}^{(i)} + \Delta_{S'_f{}^{(i)}}) \ldots (H)$$

and again, supposing the Δ small, these equations give the difference equations

$$\Delta_{S_f} = \Delta_{S_{f\,o}} + \sum_{i=h}^{i=m} \Delta_{S_i}\,S_f'^{(i)} + \sum_{i=h}^{i=m} S_i\,\Delta_{S'_f{}^{(i)}} \ldots\ldots\ldots (K)$$

Now these Equations (K), together with Equations (F), may form the basis of our solution. In them the Δ_{S_f} are the quantities sought. The $\Delta_{S_{f_o}}$ being the changes in stress of the framework composed of the non-superfluous bars, under the given loading, due to the change of its form for the joint displacements of the given indeterminate framework under that loading; and the $\Delta_{S_f{}^{(i}}$ being the corresponding changes in the coefficients $S'_f{}^{(i)}$, they are both determinable by the method for determinate frameworks, explained previously.

As to the solution of Equations (F) and (K), besides the analytic methods, we may readily apply to them any of the usual graphic methods for the determination of stresses in indeterminate frameworks. The similarity of their forms to those of the equations for

Mr. Cilley. stresses in indeterminate frameworks will suggest the manner of utilizing the same methods. Thus one may take advantage here, as in the case of the calculation of changes in stress accompanying changes of form in determinate frameworks, of all the accumulated methods and resources, graphical and analytical, now in our hands for the calculation of the ordinary stresses on the supposition of no change in form.

The writer regrets that his present limited time prevents his illustrating the use of these methods by a calculation of the additional stresses in the two-hinged arch of the paper, resulting from the flattening of that arch accompanying a fall of temperature. It is evident, however, that they would be quantities of the same proportions as those already found for the three-hinged arch.

In concluding this brief exposition of the theory of the calculation of changes in stress in frameworks accompanying small changes in form, the writer believes he can commend its use for cases where the actual changes are really quite considerable, that is to say, for the approximate calculation of the corrections to be given to the stresses in flexible frameworks, as slender arches and stiffened suspension bridges. Although the work is laborious, it is only laborious in the same way and degree as the work of the usual methods for the ordinary calculations on which it is based, and it certainly is far simpler than the application of higher mathematics called for by the present rudiments of methods, proposed or used in such cases. The somewhat sketchy character of the writer's present exposition he trusts will be pardoned as the treatment of the subject here given is wholly new, so far as the writer knows, and was developed, at very brief notice, by him, for this discussion.

Before leaving the subject of temperature stresses the writer desires to note his exception to Mr. Lindenthal's quantitative statement regarding the extra material required to meet the three-hinged arch temperature stresses in the cases of the Washington arch and of the Niagara Falls road bridge (or Clifton) arch. Such claims are too easily made to be acceptable when unaccompanied by any figures in their support, and, above all, when rendered doubtful by general analysis, as in the present case.

To Mr. Lindenthal's claim, that "a proper comparison" of the strain sheets "of the two types of arch treated by Mr. Cilley" (two and three-hinged), demands a correction "for the large bending moments resulting from the immobility of the hinges, incorrectly ignored by him," the writer feels that he must strongly take exception. Mr. Lindenthal forgets that in the paper it was particularly specified that only ideal frameworks with frictionless joints were under consideration. Moreover, the pin is not a necessary connection member for this purpose. Thin plates have not only been proposed, but also used.

In the Kaiserstег bridge* at Oberschoenweide, near Berlin, designed by Müller-Breslau, such a plate hinge was used, and in France an entire set of joints has been constructed recently with such plates, in a bridge of 132-ft. span, described in the *Annales des Ponts et Chaussées.* Mr. Cilley.

Thus, while a correction is necessary where pins are used, they are really unnecessary indeterminate adjuncts, and therefore are not properly to be considered in such a comparison as that made in the paper.

Mr. Lindenthal says that temperature stresses and hinge friction require additions to the sections of both two and three-hinged arches, which cannot be ignored, and follows this with the somewhat extreme statement—"When made (these additions), the two-hinged arch will in every case show great economy over the three-hinged arch." Again, the writer feels that we are entitled to the figures in proof of this bare assertion. The analytic considerations, thus far exposed, by no means support it.

Mr. Lindenthal holds that it "is not necessary, either for safety or economy," that the strains in an indeterminate framework "be determined with the same exactness as in statically determined structures." If Mr. Lindenthal thus unfairly favors indeterminate designs, it may well be that he will find in them the higher economy he claims, but it is certainly competent for the opponents of such designs to take exceptions to such procedure. It must be remembered that it does not take much favoring to eat up the small margin of a few per cent. which will ordinarily limit the difference in material required in the two cases. The smallness of this difference is precisely the reason why the writer argues rather the lack of economic advantage in the use of indeterminate forms, than its presence in the use of the determinate forms.

Mr. Lindenthal states that "no failures of bridges, caused by their indeterminateness, have yet occurred." Has he not, perhaps, overlooked some cases? Partial failures due to indetermination have been anything but wanting. Many a weak member has been broken or has broken its connection when carried along by the movement of some heavy member to which it was improperly attached; many a case of initial strain, permitted by the use of redundant members, has resulted in rupture; many a continuous floor system has failed in the rivets, under the stresses actually coming upon it, for which it was not designed; and in accidents like the buckling of the lower chords of the stiffening trusses of the Brooklyn bridge over the towers, we have illustrations of the consequences of indetermination in the case of more important parts. It may be said that these were due, not to the use of indeterminate construction, but to faulty calculation, or neglect to calculate at all, or to faulty adjustment. But these things arose purely from the indetermination and may therefore properly be

* Described in *The Engineering Record* of Feb. 17, 1900.

Mr. Cilley. charged against it. Far from indetermination never being a cause of failure, it has been a fruitful cause, especially in other structures than bridges, where temperature strains have worked havoc.

We now come to the last of Mr. Lindenthal's objections, which is to deflection as a criterion of rigidity. He says: "What is a rigid bridge? If we take rigidity in metallic bridges to mean absence of vibration, then deflections are a deceptive criterion."

If this were merely a question of difference of definition of the word "rigidity," it would have no further importance, but Mr. Lindenthal continues: "The public considers a bridge which vibrates, as an inferior structure; and one which does not, as a superior one. This ought to be the guide, also, to the engineer." But this argument of "*Vox populi vox Dei*" is hardly convincing, and Mr. Lindenthal's illustrations only serve to increase the writer's distrust.

Vibration, far from being an indication of anything defective, is the evidence of a highly elastic and rigid constitution. Only rigid and elastic large bodies vibrate rapidly, and therefore, in a fashion very noticeable to us. The vibrations of a slack rope are not marked, but it is scarcely rigid. A Peruvian rope suspension bridge is not notable for its vibration, but even the public would scarcely regard it as superior. Many a loose-jointed wooden structure is free from vibration, while in similar metal structures it is most marked, but the wooden structure is hardly therefore the superior one.

Wherever great rigidity, elasticity and lightness go together, vibration is likely to be noticeable. Where, on the contrary, there is loose or slack construction, imperfect elasticity and great mass, vibration will be imperceptible. The writer can scarcely believe that Mr. Lindenthal had seriously considered the bases of vibration before he thus proposed it as the criterion for distinguishing bad from good construction. Large deflections, which mean flexible construction, really are objectionable, because they interfere seriously with high speeds, and, moreover, allow of dangerous swaying in high winds. But rapid vibration is an evil affecting the imagination far more than it affects the structures. Moreover, Mr. Lindenthal's illustrations show that marked vibration occurs equally in indeterminate and determinate structures. It is, in fact, a matter largely independent of the presence or absence of superfluous bars.

In concluding this part of his reply, the writer feels justified in protesting at the statements wholly unsupported by evidence, and some of which analysis has shown to be incorrect, which characterize this part of the discussion. It was his hope that those who made counter claims to the arguments in the paper would adduce their evidence in such form as to permit of its independent consideration. In the absence of such evidence it can scarcely be expected that these

claims will receive much consideration, and, certainly, we cannot be expected to accept them. Mr. Cilley.

In replying to Professor Ritter's comments, the writer desires to express at the outset his appreciation of the very fair and reasonable position taken by him, as well as of the entire absence of any misapprehension or misconstruction of the paper in these comments. In the hands of men as able and as honest as Professor Ritter, the use of indeterminate frameworks loses many of its objections. But, since few engineers favoring indeterminate construction are equally well informed and fewer are as fair, its use is generally subject to the full force of the writer's objections.

Some of the points touched on by Professor Ritter have been considered by the writer in his reply to Mr. Lindenthal, but there remain several points which it is desirable to consider here.

The shorter unsupported lengths of the struts is an advantage which has frequently been cited in connection with lattice webbing, and as a justification of the European custom of riveting members together wherever they may cross. But it seems to be questionable whether the apparent advantage thus obtained is not illusory. As was long ago pointed out by Maurice Levy, the use of multiple systems of webbing results in a reduction of the sections of the posts which may well much more than offset the reduction of free length due to attachment at points of intersection. And this is apart from the fact that the free lengths for side deflections (deflections out of the plane of the truss) are not reduced through this arrangement in anything like the degree that the free lengths for deflections in the truss planes are reduced. But this is not all. Such attachment of points of intersection results in bending moments (when it does not result in anything more), which are rarely if ever considered, although sometimes very large. When these are taken into consideration it will be found that the consequent added secondary stresses much more than offset any advantage from shortened free length. Moreover, this practice of riveting every member to every other member wherever there is a chance becomes a wholly indiscriminate and blind action which sometimes entails serious unforeseen results, and has been responsible for no small part of the failures in indeterminate construction. The fact that not more than two passing co-planar members can be connected at the same point without entailing, in addition to bending stresses, direct axial stresses, is usually overlooked or not appreciated.

The writer's view is, that as far as possible we should design only as we can and do calculate and understand, and that the procedure he is now criticising, the consequences of which are neither calculated nor understood, should be put aside by all who prefer the guidance of their intelligence to slavish copying of ill-founded methods of the past. Some day it will be recognized that every unnecessary connec-

Mr. Cilley. tion is only an added restraint upon the free exercise of function, and that, as the writer has elsewhere pointed out, "Statical indetermination in a structure is always to be regarded as self-interference with efficiency."

In regard to the objections to the expense of hinges, constructed as at present, with pins, the writer ventures to suggest that this difficulty, as well as that of the shocks at the hinges, mentioned by Professor Ritter, may be overcome by the substitution of thin plates for pins. Flexible joints by no means depend on the use of pins for their realization, as is so commonly assumed.

Regarding discrepancies of length of members, the writer would point out the consequences in this direction which the theory of probability tells us we must expect. We know that an error ε in the length of any member of a statically determined framework, will change the distance, between any two of its joints not directly connected, by the amount $S'\varepsilon$ (where S' is the stress in the member due to a pair of equal and opposite forces of unity applied each at one of the two joints). Now suppose each bar f has a corresponding probable error ε_f in its length (which error may equally well be + or —), then the probable resultant error Δ, in the distance between the two joints, as a consequence of all these probable single errors ε_f, we know from the theory of probability will be

$$\Delta = \sqrt{\Sigma\,(S'\varepsilon)^2}$$

Let us apply this to an example:

Suppose we have a two-hinged arch of, say, 500 ft. span, and 90 ft. rise, with parallel chords 15 ft. apart, and that it is built out from the abutments to meet at the center. Suppose that we have forty panels and that the probable error of each member is only $\frac{1}{64}$ in. This is moderate for even the best of work, that is to say, where all lengths are determined by milled surfaces brought firmly into contact for riveted joints, or by carefully bored pin bearings. Very careful, repeated measurements made by the writer with an excellent steel tape, firmly clamped at one end and subject to a uniform pull by means of a spring balance at the other end, have convinced him that errors in measurement alone are likely to reach this amount. For simple riveted connections the work would be sensibly less accurate. Here it will readily be seen that the average value of S' for the chord members is about two-thirds, while that for the web members is less than one-fifteenth, but more than one-twentieth. There will be 80 chord members, which alone would cause a probable error in the closing middle chord bar of

$$\frac{2}{3}\sqrt{80} \times \frac{1}{64} = \frac{3}{32} \text{ in., about.}$$

With a simple triangular system, as we will suppose, there would also be about 80 web members, and taking for them the smaller average

value of S' as one-twentieth, the probable error in closing of the middle chord bar due to their errors would be Mr. Cilley.

$$\frac{1}{20}\sqrt{80}\times\frac{1}{64}=\frac{1}{128}\text{ in., about.}$$

The resultant probable error for both chords and webbing would be

$$\sqrt{\left(\frac{3}{32}\right)^2+\left(\frac{1}{128}\right)^2}$$

(since chord and web members are equally numerous) or practically $\frac{3}{32}$ in., as for the chords above. This gives an idea of what may be expected in the final result from a given degree of accuracy of workmanship, and so tells us what degree of accuracy it is necessary to maintain in a given case. Some such method should be used where indeterminate frameworks are constructed.

Professor Ritter states that "In general, structures with hinges (arches or cantilevers) are less stiff than those without hinges." The writer feels that this statement, reasonable as it appears at first glance, cannot, after all, be so readily accepted. It involves, first, the question of what are comparable structures; and, second, the question of what basis of judging stiffness is to be used. Confining ourselves to deflections for this latter, as the writer has done, there still remains the question of where the deflections are to be taken. It does not suffice, particularly with arches, to compare only center deflections, for the deflections of non-central points due to non-symmetric loadings are likely to be larger than those at the center, as in the writer's illustration. Such being the case, how shall we combine these? Shall we take the average of deflections at a series of given intervals, or shall we simply compare the greatest deflections wherever occurring? The writer would again refer to his remarks on page 374, in this connection.

In concluding this reply to Professor Ritter, the writer would suggest that perhaps "the extensive adherence of American engineers to stiff joints in upper chords, as well as rigid attachment of floor-beams, stringers and lateral, portal and sway bracing," is largely due to practical (shop) considerations, together with recognized objections to the use of pins in many cases, especially where adjustable members would be introduced. The abuse of adjustable members has proved to be a serious matter in this country. The writer has been informed of a case where a section of the bottom chord was relieved of its part of the stress and made slack by the over-tightening of counters. And, as is well known, small counters and their connections are often seriously overstrained in this manner.

Perhaps, when a practical system of making connections with thin plates has been developed, a matter which will require much time and experience, even with the best of good will on the part of designers,

Mr. Cilley. engineers will sufficiently appreciate its merits in eliminating secondary stresses, to give it preference over the present system of deceptive connections, whose strength is always much less than it is supposed to be, and whose additions to stiffness are only obtained at the expense of notable and unconsidered increases of stress in the main sections.

The comments of Professor Dietz are directed principally to pointing out what numerous factors enter into the final determination of the cost of a structure. To these the writer at once assents, but he discovers in them little or nothing affecting the validity of the arguments he has advanced in the paper. Most of the factors mentioned by Professor Dietz are equally factors with both determinate and indeterminate frameworks, and, therefore, do not influence their comparison, provided only that in any detailed comparison of two designs they are actually made equally factors in the two cases. As to the fact that the superstructure is only an element of the total, as far as it is of moment at all, it is a factor wholly favorable to determinate construction. Nothing like as carefully built and expensive foundations and supports are required for the determinate as for the indeterminate structures. The writer knows of no practical factor except the large (?) expense of pins which can be cited as adverse to determinate construction, and, aside from the fact previously noted, that their use is not necessary, the saving in cost of erection resulting from it may exceed the cost of manufacture.

Regarding Professor Dietz's detailed comparison of a two and three-hinged arch, the writer begs to note that the placing of the middle hinge below the top chords has very probably deprived the three-hinged arch of a very legitimate advantage it might have had. Moreover, until we know how the sections were determined we cannot know whether the comparison is really fair in this respect. These considerations affect the stated deflection also, but it is in a further respect misleading. It is simply the middle deflection and, as previously noted, very possibly, if not probably, is not the greatest deflection.

Finally, the writer is not inclined to agree at all with Professor Dietz's apparent view that only statistical comparisons of numerous complete projects can be conclusive. There is not now and there probably will not be (in the writer's opinion) within the next century, enough fairly comparable data to enable a definite conclusion to be drawn therefrom respecting the relative merits of determinate and indeterminate frameworks in general. Worked-out cases are almost invariably not fairly comparable. The writer holds that the best, the most reliable, if not the only fair basis of comparison of the relative merits of determinate and indeterminate frameworks must lie in such comparison of ideal structures as he has made. Actual or complete designs are subject to so many irrelevant variations that their comparison necessarily carries with it no general conclusion.

Mr. Cilley.

The writer desires to express his appreciation of the brief but very correct summing up of the paper included in the comments of Prof. Joseph Sohn. It is a great help to grasping the main points of such a paper, necessarily somewhat long and complex in its complete form, to have its essential points brought out in so concise and clear a restatement as in the present instance.

The writer is pleased to learn of Professor Jung's adherence to the use of determinate rather than indeterminate systems. Italy has long been a leader in the intelligent application of science to the arts, and notable for her freedom from the old dogmatic beliefs which in many other countries still so largely maintain their hold. The writer well remembers that many of the most lucid and valuable of the modern additions, both to the theory of structures and to the general theory of elasticity, have come from Italy, and believes that in the extension of the use of determinate construction and of intelligent designing Italy will again be in the front.